Solutions for the Mock Exam

These are only the sketches of the solutions. You should also know all the definitions from the example sheets.

Problem 1 []

(a) $\chi(G)$ is the smallest number of colors needed to color the vertices of G such that no two adjacent vertices are of the same color. The chromatic index $\chi'(G)$: replace "vertices" by the "edges".

(b) $\chi(K_n) = n$: n colors are enough (give to each vertex different color), we also need $\geq n$ colors since any two vertices are adjacent; $\chi(T_n) = 2$ since T_n is bipartite, and thus can take colors to be its bipartition classes; $\chi(K_n \setminus M) = \lceil n/2 \rceil$: here is a bit longish argument: if n is even, then deleting the maximum matching, we can color the vertices connected by an edge of M with the same color, moreover, using $n/2 - 1$ colors is not enough, since by the pigeonhole principle, there would be three vertices of the same color. But since we only deleted the edges of a matching, two out of these three vertices must be adjacent in $K_n \setminus M$, a contradiction. The case when n is odd is treated similarly. Also notice: $K_n \setminus M = K_n - M$ and we delete only the edges of M from K_n and not the vertices of M!

(c) Theorem of Kőnig states that for a bipartite G we have: $\chi'(G) = \Delta(G)$. Since T_n is tree, it is bipartite, and thus: $\chi'(T_n) = \Delta(T_n)$.

Problem 2 []

This was in the lecture. Don't forget to define $S_{0,0} = 1$ and $S_{0,k} = 0$ (for $k \in \mathbb{N}$).

Problem 3 []

The order in which we count the fruits doesn't matter. It is better to start with power series and argue about the mutliplication of formal power series. Let $h(x) := \sum_{n=0}^{\infty} h_n x^n$ be the generating function. Then we first claim:

$$h(x) = (1+x^2+x^4+x^6+\ldots)(1+x+x^2)(1+x^3+x^6+x^9+\ldots)(1+x) = \frac{(1+x)(1+x+x^2)}{(1-x^2)(1-x^3)}.$$

This is so since multiplying out the coefficients of the corresponding series gives us the number of particular fruits in the basket. Thus, the coefficient in front of x^n is exactly the number of different baskets of fruits we can produce. Now observe, that $1 - x^3 = (1-x)(1+x+x^2)$ and $1 - x^2 = (1-x)(1+x)$. Thus, $(1+x+x^2)$ and $(1+x)$ cancel out and we obtain:

$$h(x) = (1-x)^{-2}.$$

Applying generalized binomial theorem yields:

$$h(x) = \sum_{n=0}^{\infty} (n+1)x^n.$$

Problem 4 []

This is a former exercise. If you prove it via Inclusion-Exclusion, you should state IE first!

Problem 5 []

State theorem of Petersen and theorem of Tutte. Then prove (e.g. as in the lecture), that Tutte's condition is satisfied.

Problem 6 []

(a) Remark: $R(k)$ and $R(k,k)$ is the same: the minimum number n of vertices such that no matter how one colors the edges of K_n there is always a copy of monochromatic K_k.

(b) Either you use the bound $\lfloor e3! \rfloor + 1 = 17$ from the lecture (but then you need to prove it), or you argue as follows: fix any coloring of $E(K_{17})$ and pick any vertex, say 1. Since we used three colors and 1 has 16 neighbors, 1 is a adjacent to at least 6 vertices via the edges of the same color (say red). But then, if some of them are adjacent via a red edge we would obtain a monochromatic triangle; thus, assume that none of the edges between these 6 vertices is red, therefore, they are colored by at most two colors. Since from the lecture we know that $R(3,3) = 6$ there must still exist a monochromatic K_3.