Solutions for the Example sheet 1

**Problem 1** []

1. The first person may be sent any of the $k$ kinds of postcards. No matter which one he is sent, we may still send the second one any of the $k$ kinds, so there are $k \cdot k = k^2$ ways to send cards to the first two friends. Again, whatever they are sent, the third friend can still be sent $k$ kinds, etc. So there are $k^n$ ways to send out the cards. (This is just the number of maps $f\colon [n] \to [k]$, since postcards are distinguishable boxes and friends are distinguishable balls).

2. If they have to be sent different cards, the first person can still be sent any of the $k$ cards. But for any choice of this card, there are only $k-1$ kinds of cards left for the second person; whatever the first and the second friends receive the third one can get one of $k-2$ postcards, etc.... Thus the number of ways to send them postcards is $k(k-1)\dots(k-n+1)$ (which is, of course, 0 if $n > k$). (In other words this is the number injective maps from an $n$-set to a a $k$-set, i.e. the falling factorial $k^{\underline{n}}$ of length $n$.)

3. This is the same as the first question but we have $\binom{k}{2}$ pairs of postcards instead of $k$ postcards. Thus the result is $\binom{k}{2}^n$.

**Problem 2** []

(a) Imagine $k$ 1-euro coins in a row and suppose that the people come one by one and pick up euros as long as you allow them. Thus, we will have to say "Next please" $n-1$ times. If we determine at which points (after which coins) we say this, we uniquely determine the distribution. There are $k-1$ possible points to switch and we have to choose $n-1$ out of these. Hence the result is

$$\binom{k-1}{n-1}.$$

(Notice that this is the number of ordered $n$-partitions of $k$, since euros are nondistinguishable balls and people are distinguishable boxes and the map we are looking for is surjective.)

(b) We reduce it to the previous case by borrowing one euro from each person. If we distribute the $n+k$ euros we then have in such a way that each person gets at least one, we would then have done the same as if we had distributed the $k$ euros without this requirement. More precisely, distributions of $n+k$ euros among persons so that each one gets at least one are in one-to-one correspondence with all distributions of $n$ euros among $k$ person. Hence the answer is

$$\binom{n+k-1}{n-1}.$$

**Problem 3** []

There are 16! permutations of the letters of CHARACTERIZATION. However, not all of these give new words; in fact, in any permutation, if we exchange the three $A$'s, the two $C$'s, the two $R$'s, the two $I$s or the two $T$'s we get the same word. Thus for any permutation, there are $3! \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 96$ permutations which give the same word, so the result is

$$\frac{16!}{96}.$$

In general, given an alphabet (a set) $\Sigma = \{a_i | i \in [m]\}$ (with $m$ elements). We want to build sequences of length $N = \sum_{i=1}^{m} n_i$, where the $a_i$ appears $n_i$ times. Let $W$ be the set of all sequences $w$ of length $N$, such that $a_i$ appears in $w$ exactly $n_i$ times. Define the set

$$X := \cup_{i \in [n]} \{a_{i,j} | j \in \{1, \dots, n_i\}\}, \quad \text{where } a_{i,j}\text{s are distinct symbols.}$$

Further denote by $\mathrm{Bij}([N], X)$ the set of all bijections from $[N]$ to $X$ and let $f : \mathrm{Bij}([N], X) \to W$ be a (surjective) function which assigns to each bijection $\phi \in \mathrm{Bij}([N], X)$ a sequence $w$, by replacing $a_{i,j}$ through $a_i$. Notice that for every $w \in W$:

$$|f^{-1}(w)| = \prod_{i=1}^{m} n_i!,$$

and from $\mathrm{Bij}([N, X]) = \dot\cup_{w \in W} f^{-1}(w)$ and the rule of sum it follows: $|W| = \frac{N!}{\prod_{i=1}^{m} n_i!}$. Recall that this is called a multinomial coefficient.

**Problem 4** []

1. *First solution.* Let $M$ and $N$ be disjoint sets with $|N| = n$ and $|M| = m$. Define
$$\mathcal{A} := \{(X, Y) | X \subseteq M, Y \subseteq N, |X| + |Y| = k\}$$
and for $\ell \in [k] \cup \{0\}$:
$$\mathcal{A}_\ell := \{(X, Y) | X \subseteq M, Y \subseteq N, |X| = k - \ell, |Y| = \ell\}.$$
Notice that the product rule implies $|\mathcal{A}_\ell| = \binom{n}{\ell}\binom{m}{k-\ell}$, and the following function is easily seen to be a bijection:
$$\phi\colon \begin{cases} \binom{M \dot\cup N}{k} \to \mathcal{A} = \dot\cup_{\ell \in [k] \cup \{0\}} \mathcal{A}_\ell \\ S \mapsto (M \cap S, N \cap S). \end{cases}$$
Therefore, $|\binom{M \dot\cup N}{k}| = |\dot\cup_{\ell \in [k] \cup \{0\}} \mathcal{A}_\ell|$ and by the rule of sum, $\sum_{\ell=0}^{k} \binom{n}{\ell}\binom{m}{k-\ell} = \binom{m+n}{k}$.
*Second solution.* Consider the following polynomial identity
$$(1 + x)^n (1 + x)^m = (1 + x)^{m+n}.$$
By multiplying out the terms on both sides and comparing the coefficients of $x^k$ the desired equality follows.

2. We use $\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$, thus

$$\binom{n-1}{k} \leq \binom{n}{k} \Leftrightarrow k \leq n-k+1 \Leftrightarrow 2k \leq n+1.$$

**Problem 5** []

A chessboard $B$ of size $2^n \times 2^n$ is given where one arbitrary field is cut out. Show that one can perfectly tile the remaining fields by the figures of the form "L": $(1,1)$, $(1,2)$, $(2,1)$ (you can rotate the figures).

*To show* $\mathcal{P}(n)$*:* A chessboard $B$ of size $2^n \times 2^n$ with one arbitrary field cut out, can be perfectly tiled by figures of the form "L".

*Proof. Induction start:* $n = 1$ is easy.

*Induction step:* $\mathcal{P}(n) \implies \mathcal{P}(n+1)$*:* Let $B_{n+1}$ be a board of size $2^{n+1} \times 2^{n+1}$. $B_{n+1}$ can be cut into 4 equal sized boards $B^{(1)}, B^{(2)}, B^{(3)}, B^{(4)}$, each of size $2^n \times 2^n$. Thus, the fields $(2^n, 2^n), (2^n+1, 2^n), (2^n, 2^n+1), (2^n+1, 2^n+1)$ belong to different boards. W.l.o.g. we can assume that the field which was cut out is on the board $B^{(1)}$. Then we can put one figure in such a way that from the other three boards ($B^{(2)}$, $B^{(3)}$ and $B^{(4)}$) exactly one field is cut out too (by this figure). Namely from the set $(2^n, 2^n), (2^n+1, 2^n), (2^n, 2^n+1), (2^n+1, 2^n+1)$ (draw a picture). This way we obtain 4 boards each of size $2^n \times 2^n$ with one field cut out. By the induction hypothesis, we can tile each of them perfectly with $L$-figures. □