SOLUTIONS FOR THE EXAMPLE SHEET 10

Problem 1 []

Let G be a bipartite graph with n vertices and bipartition into A and B.

$\Longrightarrow$

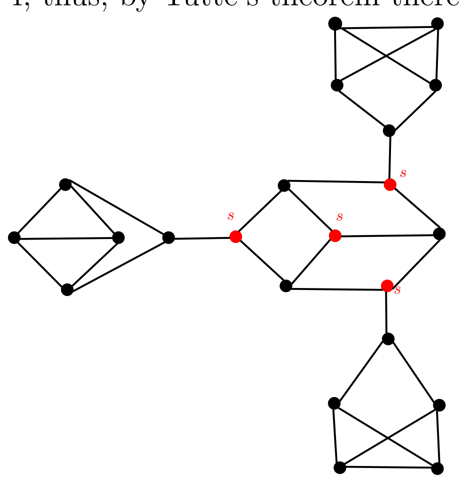
Let M be a perfect matching in G and S be the largest independent set of vertices in G. It follows that $|S| \leq n/2$, since otherwise an edge of M would lie in S. The classes of G, A and B, are themselves independent: $|A| = |B| \leq n/2$. Since $|A| + |B| = n$ and $\alpha(G) \leq n/2$ it follows that $|A| = |B| = n/2$. This gives: $\alpha(G) = n/2$.

$\Longleftarrow$

Since A and B are independent sets, it follows: $|A|, |B| \leq \alpha(G) = n/2$. Further, with $|A| + |B| = n$ we have $|A| = |B| = n/2$. We show Hall's condition. Let $S \subseteq A$. We have: $S\dot\cup(B \setminus N(S))$ is independent, implying $|S\dot\cup(B \setminus N(S))| \leq n/2$. Since $|S\dot\cup(B \setminus N(S))| = |S| + |B| - |N(S)| = |S| + n/2 - |N(S)|$, we obtain $|S| \leq |N(S)|$. Therefore, by Hall's theorem, G contains a perfect matching.

Problem 2 []

(a) Proof by picture (deleting the vertices from S we obtain $q(G - S) = 6 > |S| = 4$, thus, by Tutte's theorem there is no perfect matching):



(b) Consider a cycle of length 5, C_5:

$$\alpha'(C_5) = 2 \text{ but } \beta(C_5) = 3.$$

Therefore there exists a non-bipartite G with $\alpha'(G) < \beta(G)$.

(c) This is best shown by induction on the number of vertices for any forest. Since tree is a (connected) forest, the claim follows.
$\mathcal{P}(n)$: A forest with at most n vertices contains at most one perfect matching.
$\mathcal{P}(1)\&\mathcal{P}(2)$ are easily seen to hold.
$\mathcal{P}(n) \Longrightarrow \mathcal{P}(n+1)$: Let F be a forest with $n+1 \geq 3$ vertices. If $\Delta(F) = 0$ there is nothing to be shown. Let's assume $\Delta(F) \geq 1$. Since the components of F are trees, we can pick a vertex, say v, of degree 1. Let w be its unique neighbor in F. If M is a perfect matching of F then vw must be contained in M. Therefore,

F contains at most one perfect matching if and only if $F - \{v, w\}$ contains at most one perfect matching (which is true by the induction hypothesis since $F - \{v, w\}$ is a again a forest).

Problem 3 []

Let M be a maximal matching in G and let M_m be a matching of maximum cardinality in G. By the maximality of M, every edge from M_m matches at least one vertex in $V(M)$. And every vertex in $V(M)$ is matched by at most one edge from M_m. Therefore: $2|M| \geq |M_m|$, implying $|M| \geq \alpha'(G)/2$.

Problem 4 []

The lower bound of $5n$ edges is best possible: consider the complete bipartite graph $K_{5,n}$.

Let $G \subseteq K_{n,n}$ with $|E(G)| > 5n$. Assume that G contains no matching of size 6. By the Theorem of Kőnig and Egerváry, there is a minimum vertex cover of size at most 5. Since $\Delta(G) \leq n$, it follows that such a vertex cover can be incident to at most $5n$ edges. Since $e(G) > 5n$ we obtain a contradiction to the existence of such small cover. Therefore, there is a matching with 6 edges.

Problem 5 []

Let $G = (A, B, E)$ be a bipartite graph with $|N(S)| \geq |S|$ for all $S \subseteq A$. Let U be the minimum vertex cover of G. Consider $S := A \setminus U$. If $S = \emptyset$, then there is a matching of A in G.
Thus, assume $S \neq \emptyset$. Since U is the vertex cover, $|U \cap B| \geq |N(S)|$, which implies $|U \cap B| \geq |S|$. This in turn yields $|U| = |U \cap B| + |U \cap A| \geq |N(S)| + |U \cap A| \geq |S| + |U \cap A| = |A|$. Thus, by the Theorem of Kőnig and Egerváry, there is a matching of A in G.

Problem 6 []

Let $k \in \mathbb{N}$. Let V be a set of n elements, $k|n$, and let $V_1 \dot\cup \ldots \dot\cup V_m$ and $W_1 \dot\cup \ldots \dot\cup W_m$ be partitions of V into k-sets. We define a bipartite graph G with the vertex classes $A = \{V_1, \ldots, V_m\}$ and $B = \{W_1, \ldots, W_m\}$. Further, G has an edge V_iW_j iff $V_i \cap W_j \neq \emptyset$. If G contains a perfect matching, then we simply pick for each edge $e = V_iW_j$ a representative from $V_i \cap W_j$ (since V_is and W_js partition V).
Thus, it remains to verify Hall's condition. Let $S \subseteq A$ and let's estimate $|N(S)|$. Since each element W_j contains k elements, it follows that $|N(S)| \geq |\dot\cup_{X \in S} X|/k = |S|$ (since W_js partition V into k-sets).