Solutions for the Example sheet 11

Problem 1 []

The direction $\Rightarrow$ is clear.
$\Leftarrow$ We may assume that $|G| \geq 3$. Assume that $G = (A\dot{\cup}B, E)$ is not 2-connected and G is k-regular ($k \geq 2$). W.l.o.g. let $a \in A$ be such that $G - a$ is disconnected. Since G was connected, $k \geq 2$ and thus: $\delta(G - a) \geq k - 1$, implying that every component has at least one edge. Further, let $C = (A_c\dot{\cup}B_c, E_c)$ be a component of $G - a$. Since every vertex from $A \setminus \{a\}$ has degree k in $G - a$, we infer $|E_c| = k|A_c|$. On the other hand, some edges must have been deleted between a and B, which implies that $|B_c| > |A_c|$. Since G is k-regular and bipartite, it follows by a lemma from the lecture that G contains a perfect matching. This in turn implies that $|B_c| \leq |A_c| + 1$ as otherwise this would contradict that C is a component of $G - a$. Thus, we have $|B_c| = |A_c| + 1$. Since G is k-regular and $|E_c| = k|A_c|$, it follows that all neighbours of a must lie in B_c. But then, any component of $G - a$, other than C, must be k-regular, which contradicts that G is connected.

Problem 2 []

Partition $V(G)$ into two sets X, Y. The edges from $E(X, Y)$ form a bipartite graph H with partition $X\dot{\cup}Y$. If H contains fewer than half the edges of G incident to a vertex v, then v has more edges to vertices in its own class than in the other class. We move v to the other class, which increases the number of edges of a new created bipartite graph. We continue the same procedure with this newly obtained graph. Since the graph G is finite and at each step we increase the number of edges in a bipartite graph, the process ends.
At the end we have some biopartite subgraph H' of G such that $\deg_{H'}(v) \geq \deg_G(v)/2$ for every $v \in V(H') = V(G)$. We sum up and obtain $2e(H') \geq e(G)$ and we are done.

Problem 3 []

(a) Since $E \neq \emptyset$, $\chi(G) \geq 2$. Color the vertices of G with $\chi(G)$ colors. Set V_1 to be one of the color classes. Clearly, $\chi(G[V_1]) + \chi(G[V \setminus V_1]) = 1 + \chi(G[V \setminus V_1]) = \chi(G)$, since otherwise we could have colored G with fewer than $\chi(G)$ colors.

(b) Consider a maximal complete subgraph H of G. Assume, $H \cong K_k$. Further assume that $\chi(H) + \chi(G - V(H)) = \chi(G)$. Since $H \neq G$, we infer that $\chi(G - V(H)) = \chi(G) - k =: t$. Fix such a coloring of $G - V(H)$ and let $V_1, \ldots, V_t$ be the color classes of $G - V(H)$. Next we color the vertices of H by k genuinely new colors. Now we recolor the vertices in V_1. This can be done, since V_1 is independent and every vertex from V_1 is not adjacent to some vertex in H (so we use that particular color). In this way we have shown that $\chi(G) \leq k + t - 1 < \chi(G)$ a contradiction. Thus, $\chi(H) + \chi(G - V(H)) > \chi(G)$.

Problem 4 []

(a) We show this by induction on the number k of classes. For $k = 1$, there is nothing to be shown. Let $V_1\dot{\cup}\ldots, V_k$ be a partition of $V(G)$ with the property described in the statement. By the induction hypothesis, $\chi(G-V_k) \leq |G| - |V_k| - k + 2$. Let c be the coloring of $V(G) - V_k$ with colors $1, \ldots, |G| - |V_k| - k + 2$. We color the vertices of V_k with colors $|G| - |V_k| - k + 3, \ldots, |G| - k + 2$.

Next we will recolor some vertices and reduce the number of colors by one. We know by assumption that each V_i ($i \in [k-1]$) contains a vertex v_i which is not adjacent to some vertex $w_i \in V_k$. Since we used $|G|-|V_k|-k+2$ colors to color the vertices $V(G) \setminus V_k$, one of the colors is used to only color the vertices in $\{v_1, \ldots, v_{k-1}\}$ (pigeonhole principle). We can now recolor the vertices v_i of that color by the colors of the corresponding vertices w_i. In this way we have shown:

$$\chi(G) \leq |G| - k + 1.$$

(b) We first show that $\chi(G)+\chi(\overline{G}) \leq |G|+1$. For this we take a coloring of $V(G)$ with $\chi(G)$ many colors thus obtaining a partition of $V(G)$ into nonempty sets $V_1, \ldots, V_{\chi(G)}$. Since there is an edge between any two color classes, we know by (a), that $\overline{G}$ satisfies: $\chi(\overline{G}) \leq |G| - \chi(G) + 1$. Adding $\chi(G)$, we are done.

Next we show $\chi(G)\chi(\overline{G}) \geq |G|$. Indeed, consider a coloring of G with $\chi(G)$ many colors. Since each color class is independent, we have

$$\alpha(G) \geq |G|/\chi(G).$$

And thus, the complement of G contains a complete graph on $\alpha(G)$ vertices implying: $\chi(\overline{G}) \geq \alpha(G)$. We are done.

Problem 5 []

(a) $\Longleftarrow$: follows trivially: deleting any vertex from G, we still have "another" path between any two vertices.
$\Longrightarrow$: Let G be a 2-connected graph on at least three vertices and let H_0, H_1,..., H_t be a series of the 2-connected graphs from the ear decomposition theorem (see the lecture). Recall, that H_i arises by adding a new H_{i-1}-path to H_{i-1}. The claim now follows from the fact that between any two distinct vertices in *any* H_i there are two independent paths. Indeed, by the definition, H_0 is a cycle and the claim is easily seen to be true (fix any two vertices, the cycle consits of two independent paths between them). Next assume that the claim holds for H_{i-1}.

Now we add an H_{i-1} path P and form H_i. Further assume P has as ends the vertices a and b say. Now let u, v be two vertices from $V(H_i)$ and we need to show that there are two independent u-v-paths in H_i.

If u, $v \in V(H_{i-1})$ then this is surely true. Next suppose that u, $v \in V(P)$. Here we clearly may use as one path a subpath of P, and for the other path we go from u to (w.l.o.g.) a say, then from a to b in H_{i-1}, then from b to v along the path P.

Next assume that $u \in V(H_{i-1})$ and $v \in V(P)$. By the property of H_{i-1}, there are two independent a-u-paths in H_{i-1} (say P_1 and P_2). Further, if b lies on one of these paths, then we are done, since we could use only the path from v to b and another one. Now we just extend the paths to two disjoint v-u-paths by using the edges of P. Thus, our assumption now is that b doesn't lie on these paths. We can delete a from $V(H_{i-1})$ and there is still a path that connects b to a vertex of $V(P_1) \cup V(P_2)$ and doesn't use any further vertices from $V(P_1) \cup V(P_2)$. Assume that there is an independent path from b to some vertex x of P_1, say. Then we change the path of P_1 as follows: we start from u and follow P_1 until we reach x and then we follow the path to b. Now we have to independent paths one from u to a and the other from u to b. Since $v \in V(P)$ and P is the H_{i-1}-path, we can extend these paths to two independent u-v-paths in H_i.

(b) We construct an auxiliary graph G' as follows. We add two new vertices to G', a and b say, and connect them to the ends of the edges e_1 and e_2. This G' remains 2 connected as well (easy to check). Next, (a) asserts that there are two independent a-b-paths in G'. These two paths form a cycle C in G'. Deleting from C the vertices a and b and adding the edges e_1 and e_2 we obtain the desired cycle in G (observe: a was connected only to the ends of e_1 and b to the ends of e_2).

Problem 6 []

(a) Since any maximal graph without a cutvertex on at least 3 vertices is 2-connected (see the definition of 2-connectedness), it follows that if a block is not a maximal 2-connected subgraph of G, then it is either a single (isolated) vertex or a bridge (otherwise an edge would lie on a cycle and, therefore, the block would have more than 2 vertices).

(b) Suppose that two blocks intersect in more than one vertex of G. Then their union is 2-connected as well, since deleting one vertex, both blocks remain connected and still share a vertex. Furthermore, if two blocks intersect in a vertex v, then v must be a cutvertex in G since otherwise the blocks would still be connected in G, even after deleting v (contradicting the definition of a block).

(c) Assume now that $B(G)$ contains a cycle C. Further we assume that C consists of blocks $B_1, \ldots, B_t$ (which are traversed in that order). We claim that the union of the blocks (call this graph H) from C is again a block (which would be a contradiction). Indeed, delete an arbitrary vertex x from H. And let a, $b \in V(H) \setminus \{x\}$. Assume that x was deleted from the block B_1 and was not the cutvertex from $V(B_1) \cap V(B_2)$(w.l.o.g.). Further assume that $a \in B_i$ and $b \in B_j$ $(i \leq j)$. And let b_is be the cutvertices from $V(B_i) \cap V(B_{i+1})$ $(i < t)$.

Then, a is connected to b_i, b_i is connected to b_{i+1}, ..., b_{j-1} is connected to b. Thus, a is still connected to b.
Thus, we have shown that $B(G)$ is a forest.

(d) If $B(G)$ is a tree, then picking any two vertices from G, finding the path in $B(G)$ between their blocks and then finding paths within the blocks and composing them to a path in G shows that G is connected. On the other side, if G is connected, then we can construct a $B(G)$ and to connect any two blocks we simply find first a path between any two vertices from this block. By following this path we obtain the path in $B(G)$. The cases of two cutvertices or a cutvertex and a block are treated similarly.

If B_1, ..., B_t are the blocks of G, then

$$\chi(G) = \max_{i \in [t]} \chi(B_i).$$

Indeed, since $B(G)$ is a tree, we can order its blocks and cutvertices of G in such a way that each vertex of $B(G)$ is connected to exactly one vertex from $B(G)$ to the left (this was a general proposition about the vertex ordering of a tree). Next we start coloring vertices of G, according to which block/vertex they belong to. At any time, since the blocks intersect it exactly one vertex and we have $\max_{i \in [t]} \chi(B_i)$ at our disposal, we can complete the coloring.