Solutions for the Example sheet 12

Problem 1 []

Let $G = (V, E)$ be a graph with n vertices. We label the vertices $1, \ldots, n$. Further we define an embedding ϕ by assigning to the vertex i the point (i, i^2, i^3) in $\mathbb{R}^3$. We draw an edge ij as the straight line segment

$$\{(1-\lambda)(i, i^2, i^3) + \lambda(j, j^2, j^3) \colon \lambda \in [0, 1]\}.$$

Observe that these line segments do not cross. Otherwise, there would exist four vertices i, j ,k, $\ell \in [n]$ and $\lambda, \mu \in (0, 1)$ such that

$$(1-\mu)(i, i^2, i^3) + \mu(\ell, \ell^2, \ell^3) = (1-\lambda)(j, j^2, j^3) + \lambda(k, k^2, k^3). \tag{1}$$

Since $(1-\mu) + \mu = (1-\lambda) + \lambda$, it follows that:

$$(1-\mu)(1, i, i^2, i^3) + \mu(1, \ell, \ell^2, \ell^3) - (1-\lambda)(1, j, j^2, j^3) - \lambda(1, k, k^2, k^3) = (0, 0, 0, 0). \tag{2}$$

If all i, j, k, ℓ distinct, then since the scalars in front of the $\phi(i)$, $\phi(j)$, $\phi(k)$ and $\phi(\ell)$ are not all equal to zero, we obtain a contradiction because $\phi(i)$, $\phi(j)$, $\phi(k)$ and $\phi(\ell)$ are the row vectors of Vandermonde matrix, and thus linearly independent. Assume that $i = j$, but then we obtain:

$$(\lambda - \mu)(1, i, i^2, i^3) + \mu(1, \ell, \ell^2, \ell^3) - \lambda(1, k, k^2, k^3) = (0, 0, 0, 0),$$

implying again, by the linear independence, that $\lambda = \mu = 0$, thus the straight line segments meet in one of their endpoints, which is fine since the corresponding edges then have a common endpoint.

Problem 2 []

(a) K_5 is not planar, since $e(K_5) = 10$, $v(K_5) = 5$, but $3 \cdot 5 - 6 < 10$, contradicting the bound on the number of edges in a planar graph.

(b) Consider any drawing of $K_{3,3}$ in the plane. Assume that $K_{3,3}$ is planar. Then, fix any planar drawing of $K_{3,3}$ in $\mathbb{R}^2$. In particular, the arc representing edges of some cycle of length 6 form a curve C. However, there are three other arcs corresponding to the edges connecting the vertices of this cycle. Furthermore, the ends of any two of these arcs occur in alternating order on the curve C. Thus, some two of them must cross when they go through the same region of $\mathbb{R}^2 \setminus C$, which is a contradiction. Therefore, $K_{3,3}$ is not planar.

(b') An alternative solution would be the following. Show first using the Euler's formula, that any planar drawing of a maximal planar graph G which contains no K_3 as a subgraph, has all its faces bounded by the cycles of length 4. This gives then the estimate $4f(G) \leq 2e(G)$, thus: $v(G) - e(G) + 2e(G)/4 \geq 2$ implying $e(G) \leq 2v(G) - 4$ for triangle-free planar graphs. This inequality is violated by $K_{3,3}$. Thus, $K_{3,3}$ is not planar.

Problem 3 []

We define G as follows. First we take $n \geq 3$ vertex-disjoint paths P_2 of length 2. Now we fix for each path one of its endvertices and we identify these obtaining G. Clearly, G is connected. Formally:

$$V(G) = \{0, 1, \ldots, 2n\} \quad \text{and} \quad E(G) = \{\{0, 2i-1\} : i \in [n]\} \cup \{\{i, 2i\} : i \in [n]\}.$$

Next we claim that its square G^2 is not hamiltonian. In what follows it is instructive to draw a small picture, say for $n = 4$, to follow the argument. Suppose the contrary and that C is a Hamilton cycle in G. Let's follow C starting at the vertex 0. Then there are two choices:

1. we move from 0 to an even-numbered vertex, say 2. But since the degrees all even-numbered vertices are 2 in G^2, we have to move immediately thereafter to its odd-numbered neighbor 1. From this neighbor we can only move to another odd-numbered vertex, say 3, (otherwise we would complete a too short cycle K_3). But then if we move from 3 to 0 or 4 (its only even neughbors) we again will end up in a situation where we have gone along a cycle of length at most 5 (but C has $2n + 1 \geq 7$ vertices). Thus, we move from 3 to yet another odd-numbered vertex, but then we won't be able to visit the vertex 4, which is a contradiction that C is a Hamilton cycle.

2. Thus, we have to move from 0 to some odd-numbered vertex, say 1. But then we cannot move from 1 to 2 since otherwise we complete a cycle K_3. Thus, we move from 1 to another odd-numbered vertex, meaning that we won't be able to visit 2(similarly as in the case 1). A contradiction.

Problem 4 []

First observe that G as connected graph contains a spanning tree. It is sufficient to prove that T^3 any tree T with at least 3 vertices We show the following statement $\mathcal{A}(\mathbf{n})$, which will then imply that G^3 is hamiltonian: namely, the ordering of the vertices in the statement $\mathcal{A}(\mathbf{n})$ below gives a Hamilton cycle.

$\mathcal{A}(\mathbf{n})$: For every tree T with $n' \leq n$ vertices the following is true. For every edge $vv' \in E(T)$ there is an ordering $v_1, v_2, \ldots, v_{n'}$ of the vertices such that $v_1 = v$, $v_{n'} = v'$ and the distance between v_i und v_{i+1} in T is at most 3 (for $i = 1, 2, \ldots, n' - 1$).

Start: For $n = 1, 2$ we the statement obviously holds.

$\mathcal{A}(\mathbf{n}) \Longrightarrow \mathcal{A}(\mathbf{n+1})$: Let T_{n+1} be a tree with $n+1$ vertices and let $vv' \in E(T_{n+1})$. Further: $T_{n+1} - vv'$ is a forest consisting of exactly two trees. Denote these by T and T' ($v \in V(T)$, $v' \in V(T')$). Let a be a neighbor of v in T and b a neighbor of v' in T' (if one of the trees consists of a single vertex then assume that it is its neighbor). Thus we have: $av \in E(T)$, $bv' \in E(T')$. Since $|V(T)|, |V(T')| \leq n$, we can order the vertices of T and T', by the inductive assumption $\mathcal{A}(\mathbf{n})$. Let $w_1, \ldots, w_{|T|}$ be the ordering of $V(T)$ with $w_1 = v$ and $w_{|T|} = a$. Let $u_1, \ldots, u_{|T'|}$ be the ordering of $V(T')$ with $u_1 = b$ and $u_{|T'|} = v'$. Together, the ordering $w_1, \ldots, w_{|T|}, u_1, \ldots, u_{|T'|}$ satisfies $\mathcal{A}(\mathbf{n+1})$, since the distance between a and b in T is at most 3.

Problem 5 []

This is Theorem 2.3 in "Graph Theory" book by Bondy and Murty (can be found in the library). A hint: use induction on the number n of vertices. Try to argue what happens if you cannot extend a path of length n to a (directed) path of length $n + 1$ (how is the vertex not on the path connected to the other vertices?).

Problem 6 []

This is Theorem 7.1.16 from "Introduction to Graph Theory" by D.West (can be found in the library – Handapparat Prof. Szabó). The two nonisomorphic graphs are K_3 and $K_{1,3}$.