Solutions for the Example sheet 2

Problem 1 [Bell numbers]

(a) Let's rewrite the sum:

$$\sum_{k=0}^{n} \binom{n}{k} B_k = \sum_{k=0}^{n} \binom{n}{n-k} B_k$$

Let $\mathcal{B}_S$ the set of all partitions of an i-set S, where $\mathcal{B}_\emptyset := \{\emptyset\}$ (i.e. $|\mathcal{B}_\emptyset| = 1$), and therefore we have for all $i \in [n+1] \cup \{0\}$ that $|\mathcal{B}_i| = B_i$.

We define a following mapping

$$\varphi\colon \mathcal{B}_{n+1} \to \dot{\bigcup}_{i=0}^{n} \dot{\bigcup}_{A \in \binom{[n]}{n-i}} \mathcal{B}_{[n]\setminus A} \times \{A \cup \{n+1\}\}$$

as follows. Let $k \in [n] \cup \{0\}$ and $P \in \mathcal{B}_{n+1}$, then there exists a subset A of $[n+1]$ containing $n+1$ and such that A is a block in P. We denote all blocks from $P \setminus \{A\}$ through $P' := P \setminus \{A\}$, these contain altogether k elements (thus, P' is a partition of k elements). Finally, we set $\varphi(P) = (P', A)$

Conversely, we assign to each pair (P', A) with $|A| = n-k+1$ and $(n+1) \in A$ and P' a partition of k elements the following partition of $[n+1]$: $\{A\} \cup P'$. Thus, φ is a bijection and it holds:

$$B_{n+1} = \sum_{i=0}^{n} \sum_{A \in \binom{[n]}{n-i}} |B_{[n]\setminus A}| = \sum_{i=0}^{n} \sum_{A \in \binom{[n]}{n-i}} B_i = \sum_{i=0}^{n} \binom{n}{n-i} B_i = \sum_{i=0}^{n} \binom{n}{i} B_i.$$

The same solution in a short form: Fix $a \in N$, where N is an $(n+1)$-set. Further we classify partitions of N according to the size of the block that contains a. Indeed, the number of partitions of N with the block of size $k+1$ that contains a is $\binom{n}{k} B_{n-k}$, since there are $\binom{n}{k}$ ways to choose the other k elements for the block containing a and the remaining $(n-k)$ elements of N can be partitioned arbitrarily. By the rule of sum (using $\binom{n}{k} = \binom{n}{n-k}$ and after reindexing) we get:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} = \sum_{k=0}^{n} \binom{n}{n-k} B_{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

(b) We prove the formula $B_n = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^n}{i!}$ by induction on n.

Proof.
Induction hypothesis $\mathcal{P}(n)$: for every $m \in \{0, 1, \ldots, n\}$ it is the case that $B_m = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^m}{i!}$.
Induction start: $B_0 = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^0}{i!} = \frac{1}{e} \sum_{i=0}^{\infty} \frac{1}{i!} = \frac{e}{e} = 1$.
Induction step: We use the identity from (a)

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

Now we can use the induction hypothesis and for every $k \in [n] \cup \{0\}$ for B_k we substitute the sum $\frac{1}{e}\sum_{i=0}^{\infty}\frac{i^k}{i!}$:

$$B_{n+1} = \sum_{k=0}^{n}\binom{n}{k}\frac{1}{e}\sum_{i=0}^{\infty}\frac{i^k}{i!},$$

since the series are absolutely convergent we may interchange the summation ordering:

$$B_{n+1} = \frac{1}{e}\sum_{i=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{i^k}{i!} = \frac{1}{e}\sum_{i=0}^{\infty}\frac{(i+1)^n}{i!} = \frac{1}{e}\sum_{i=0}^{\infty}\frac{(i+1)^{n+1}}{(i+1)!} = \frac{1}{e}\sum_{i=1}^{\infty}\frac{(i)^{n+1}}{i!} = \frac{1}{e}\sum_{i=0}^{\infty}\frac{(i)^{n+1}}{i!}.$$

□

Problem 2 []

(a) Let's count the pairs of sets (A, B) with $B \subseteq A \subseteq [n]$, $|B| = k$ and $|A| = m$ $(n \geq m \geq k \geq 0)$.
One way: we first choose the set A in $\binom{n}{m}$ ways and then we choose within the set A a k-set B in $\binom{m}{k}$ ways giving the total of $\binom{n}{m}\binom{m}{k}$ pairs.
The other way: we first choose a k-set B from $[n]$ in $\binom{n}{k}$ and then we can choose an m-set A, $A \supseteq B$, in $\binom{n-k}{m-k}$ ways (since $A \setminus B \subseteq [n] \setminus B$). This gives the total of $\binom{n}{k}\binom{n-k}{m-k}$ pairs. Therefore,

$$\binom{n}{k}\binom{n-k}{m-k} = \binom{n}{m}\binom{m}{k}$$

Finally we see using $\sum_{k=0}^{m}\binom{m}{k} = 2^m$ that

$$\sum_{k=0}^{m}\binom{n}{k}\binom{n-k}{m-k} = \sum_{k=0}^{m}\binom{n}{m}\binom{m}{k} = 2^m\binom{n}{m}.$$

(b) We perform the following chain of computations:

$$\binom{2n}{2k}\binom{2n-2k}{n-k}\binom{2k}{k} = \binom{2n}{2n-2k}\binom{2n-2k}{n-k}\binom{2k}{k} = \binom{2n}{n-k}\binom{n+k}{n-k}\binom{2k}{k} = \binom{2n}{n-k}\binom{n+k}{2k}\binom{2k}{k} = \binom{2n}{n-k}\binom{n+k}{k}\binom{n}{k} = \binom{2n}{n+k}\binom{n+k}{n}\binom{n}{k} = \binom{2n}{n}\binom{n}{k}\binom{n}{k} = \binom{2n}{n}\binom{n}{k}^2.$$

Problem 3 []

Let $\mathcal{A}$ be the set of k-subsets of $[n]$, which contain no two consecutive numbers. Choose $A \in \mathcal{A}$ and let $a_1 < a_2 < \ldots < a_k$ be the elements of A. Define $\phi(A) := \{a_1, a_2 - 1, a_3 - 2, \ldots, a_k - (k-1)\}$ and notice that $\phi(A)$ is also a k-element set. Indeed, from $A \in \mathcal{A}$ it follows $a_i + 2 \leq a_{i+1}$ for $i \in [k-1]$, and therefore it holds $a_i - (i-1) < a_{i+1} - i$ for all $i \in [k-1]$).
The mapping $\phi : \mathcal{A} \to \binom{[n-k+1]}{k}, A \mapsto \phi(A)$ is easily seen to be bijective (its inverse is $\phi^{-1}(a_1 < a_2 < \ldots < a_k) = (a_1, a_2 + 1, \ldots, a_k + k - 1))$ and therefore we obtain $|\mathcal{A}| = \binom{n-k+1}{k}$.

Problem 4 []

We can distribute the first kind of postcards in

$$\binom{a_1 + n - 1}{n - 1}$$

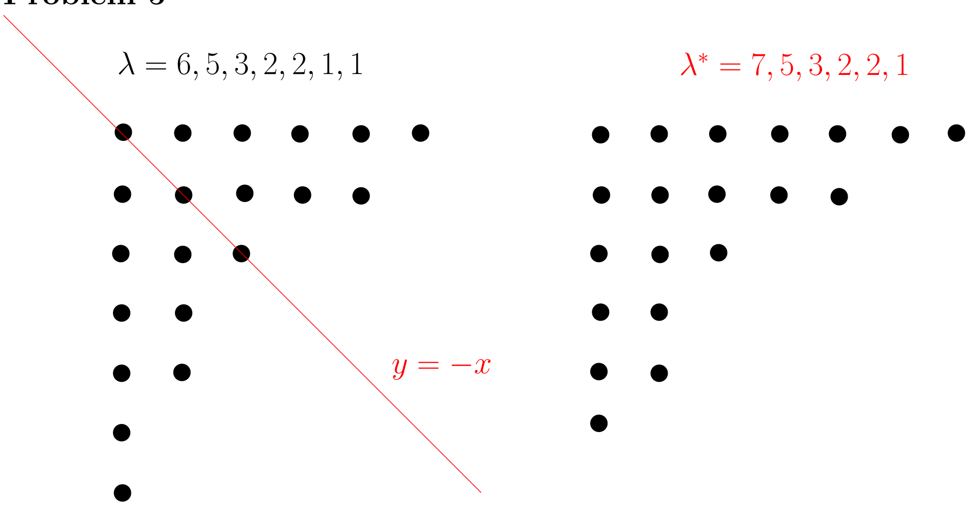
ways by the solution to problem 2(a) from the previous sheet. Whatever the choice here, there are

$$\binom{a_2 + n - 1}{n - 1}$$

ways to send out the second one, etc. Thus, the result is

$$\prod_{i \in [k]} \binom{a_i + n - 1}{n - 1}.$$

Problem 5 []



(a) let $\lambda = \lambda_1 \lambda_2 \ldots \lambda_k$ be a number-partition of n into k parts ($k \leq r$) (recall $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 1$). We can construct a diagram (a so-called Ferrer diagram) of the partition as follows. We put λ_1 dots in row 1, λ_2 dots in row 2 and so on, starting in the same column (see the picture). Now, reflecting the diagram at the main diagonal ($y = -x$) we obtain another partition λ^* with each part of size at most k. One can easily see that this reflection is a bijection between the set of number-partitions of n into no more than r terms and the set of number-partitions of n into any number terms, each of size at most r.

(b) Recall that $P(n, m)$ denotes the number-partitions of n into m parts. Let $P(n, \leq m)$ denote the number-partitions of n into at most m parts.

   (ii) Consider the mapping $\phi \colon P(n, m) \rightarrow P(n - m, \leq m)$ with $\phi(\lambda_1, \lambda_2, \ldots, \lambda_k) = \lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_m - 1$ (where we omit 0s). Clearly, ϕ is a bijection (we obtain its inverse by adding back 1s), therefore, the first claim follows.

   (i) Now let $P_d(n, m)$ denote the number-partitions of n into exactly m *distinct* parts. We define the map $\psi \colon P_d(n, m) \rightarrow P_d(n - \binom{m}{2}, m)$ by $\psi(\lambda_1, \lambda_2 \ldots, \lambda_{m-1}, \lambda_m) = (\lambda_1 - (m-1)), (\lambda_2 - (m-2)) \ldots, (\lambda_{m-1} - 1)), \lambda_m$. Again, ψ is a bijection, whose inverse can be obtained by adding back numbers $(m - 1)$, $(m - 2)$, $\ldots$, 0 to the parts of a partition from $P_d(n - \binom{m}{2}, m)$.

   Thus, the number of partitions of n into exactly m distinct parts is equal to the number of partitions of $n - \binom{m}{2}$ into exactly m parts.