Solutions for the Example sheet 3

Problem 1 []

(a) Let A be the set of the positive integers less than 1000 which are not divisible through any of the numbers 2, 3, ..., 9, i.e.

$$A := \{n \mid n < 1000, i \not| n, i \in [2,9]\}.$$

We define the sets $A_i := \{n \mid n < 1000, i|n\}$, where $i \in [9] \setminus \{1\}$. We further have: $A = [999] \setminus \cup_{i=2}^{9} A_i = [999] \setminus (A_2 \cup A_3 \cup A_5 \cup A_7)$, since $A_4, A_6, A_8 \subset A_2$ und $A_9 \subset A_3$. We use inclusion-exclusion principle to show:

$$|A| = 999 - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + |A_2 \cap A_3 \cap A_5 \cap A_7| =$$
$$999 - (499+333+199+142) + (166+99+71+66+47+28) - (33+23+14+9) + 4$$
$$= 228,$$

where for $I \subseteq \{2,3,5,7\}$ holds: $|\cap_{i \in I} A_i| = \left\lfloor \frac{999}{\prod_{i \in I} i} \right\rfloor$.

(b) Define the set B as those permutations that do not map any even number to itself:

$$B := \{\tau \colon \tau \in S_{10}, \tau(2i) \neq 2i \ \forall i \in [5]\}.$$

Further let $B_i := \{\tau \colon \tau \in S_{10}, \tau(2i) = 2i\}$, for every $i \in [5]$,i.e. B_i consists of those permutations from S_{10} that fix $2i$. Again, the principle of inclusion-exclusion yields:

$$|B| = \sum_{J \subset [5]} (-1)^{|J|} |\cap_{j \in J} B_j| = \sum_{J \subset [5]} (-1)^{|J|} (10-j)! = \sum_{j=0}^{5} (-1)^j \binom{5}{j} (10-j)! =$$
$$10! - 5 \cdot 9! + 10 \cdot 8! - 10 \cdot 7! + 5 \cdot 6! - 5! = 2170680.$$

where for $J \subseteq \{1,2,3,4,5\}$ we have $|\cap_{j \in J} B_j| = (10-j)!$, since after fixing the images of j elements, we can extend this to a permutation in exactly $(10-j)!$ ways.

Problem 2 []

First we prove by induction on n that

$$\mathcal{P}(n)\colon \quad \sum_{k=0}^{n} S_{n+1,k+1} x^{\underline{k}} = (x+1)^n. \tag{1}$$

Proof.
Induction start $(n=0)$**:** $1 = 1$.

Induction step ($\mathcal{P}(n) \implies \mathcal{P}(n+1)$)**:**

$$(x+1)^{(n+1)} \overset{\mathcal{P}(n)}{=} (x+1)\sum_{k=0}^{n} S_{n+1,k+1} x^{\underline{k}} =$$
$$\sum_{k=0}^{n} S_{n+1,k+1} x^{\underline{k+1}} + k\sum_{k=0}^{n} S_{n+1,k+1} x^{\underline{k}} + \sum_{k=0}^{n} S_{n+1,k+1} x^{\underline{k}} =$$
$$\sum_{k=1}^{n+1} S_{n+1,k} x^{\underline{k}} + \sum_{k=0}^{n} (k+1) S_{n+1,k+1} x^{\underline{k}} = \sum_{k=0}^{n+1} S_{n+2,k+1} x^{\underline{k}},$$

where we use the recurrence relation for Stirling numbers of the second kind:

$$S_{n+2,k+1} = S_{n+1,k} + (k+1) S_{n+1,k+1}.$$

□

Recall that $x^n = \sum_{k=0}^{n} S_{n,k} x^{\underline{k}}$. Thus, if we multiply out $(x+1)^n$ and replace x^n by $\sum_{k=0}^{n} S_{n,k} x^{\underline{k}}$ in (1), we obtain:

$$\sum_{k=0}^{n} S_{n+1,k+1} x^{\underline{k}} = \sum_{m=0}^{n} \binom{n}{m} x^m = \sum_{m=0}^{n} \binom{n}{m} \sum_{i=0}^{m} S_{m,i} x^{\underline{i}},$$

and by comparing the coefficients in front of $x^{\underline{k}}$ (recall that $x^{\underline{k}}$s form a basis), we get:

$$S_{n+1,k+1} = \sum_{m=0}^{n} \binom{n}{m} S_{m,k}. \tag{2}$$

We can also verify (2) by using the same function as in Problem 1(a) of the Example sheet 2. Since $B_n = \sum_{i=0}^{n} S_{n,i}$ we use (2) and sum over all $k \in \{0, 1, \ldots, n\}$ ($n \in \mathbb{N}_0$ and $m \leq n$):

$$B_{n+1} = \sum_{k=0}^{n} S_{n+1,k+1} = \sum_{k=0}^{n} \sum_{m=0}^{n} \binom{n}{m} S_{m,k} = \sum_{m=0}^{n} \binom{n}{m} B_m.$$

Problem 3 []

(a) Denote by $\mathcal{Z}_n$ the set of all number-partitions of n, i.e. those tuples $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$ with $\sum_{i \in [k]} a_i = n$ for some k. Further define two sets $\mathcal{O}_n$ and $\mathcal{N}_n$ as follows:

$$\mathcal{O}_n := \{\mathbf{a} \in \mathcal{Z}_n | \text{ all } a_i\text{s are odd}\} \quad \text{und}$$
$$\mathcal{N}_n := \{\mathbf{a} \in \mathcal{Z}_n | \text{ all } a_i\text{s are pairwise distinct}\}.$$

Additionally, let E_is be the following sets:

$$E_i := \{\mathbf{a} | \mathbf{a} \in \mathcal{Z}_n, \mathbf{a} \text{ contains } 2i\}.$$

Clearly, it is the case that:

$$\mathcal{O}_n = \mathcal{Z} \setminus \left(\cup_{i=1}^{\lfloor \frac{n}{2} \rfloor} E_i \right).$$

Thus, the inclusion-exclusion principle gives:

$$|\mathcal{O}_n| = |\mathcal{Z}_n| + \sum_{\emptyset \neq I \subseteq [\lfloor \frac{n}{2} \rfloor]} (-1)^{|I|} | \cap_{i \in I} E_i|. \tag{3}$$

For $\mathcal{N}_n$ we introduce further sets F_i ($i \in [\lfloor \frac{n}{2} \rfloor]$) as follows:

$$F_i := \{\mathbf{a} | \mathbf{a} \in \mathcal{Z}_n, \mathbf{a} \text{ contains as elements at least twice } i\}.$$

Similarly as before, it holds for $\mathcal{N}_n$:

$$\mathcal{N}_n = \mathcal{Z} \setminus \left(\cup_{i=1}^{\lfloor \frac{n}{2} \rfloor} F_i \right),$$

and again by the inclusion-exclusion

$$|\mathcal{N}_n| = |\mathcal{Z}_n| + \sum_{\emptyset \neq I \subseteq [\lfloor \frac{n}{2} \rfloor]} (-1)^{|J|} | \cap_{j \in J} F_j|. \tag{4}$$

By the definition of E_is it follows immediately that there is a one-to-one correspondence between $\cap_{i \in I} E_i$ and the number-partitions of $n - \sum_{i \in I} 2i$. Similarly for F_js, there is a one-to-one correspondence between $\cap_{j \in J} F_j$ and the number-partitions of $n - \sum_{i \in J} 2j$. Therefore, for $I = J$ we have

$$| \cap_{j \in J} F_j| = | \cap_{i \in I} E_i|.$$

Now by (3) and (4) we immediately obtain $|\mathcal{O}_n| = |\mathcal{N}_n|$.

(b) If $n = \lambda_1 + \ldots, \lambda_k$, $\lambda_1 > \ldots > \lambda_k \geq 1$, then set

$$\lambda_i = a_i 2^{t_i}, \text{ where } a_i \text{ is odd.} \tag{5}$$

Replacing now each λ_i by 2^{t_i} a_is in (5) we get (after rearranging the terms if necessary) a number-partition of n into odd terms (λ'_j). The number of occurrencies of the odd number λ'_j is

$$\sum_{a_i = \lambda'_j} 2^{t_i}.$$

Next we argue that the correspondence above is a bijection.
injectivity: Let λ and β be two different number-partitions of n into distinct terms (recall $\lambda = \lambda_1 \ldots \lambda_q$, $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_q \geq 1$ and $\sum_i \lambda_i = n$). Consider the number-partitions into odd numbers after applying the correspondence above (call them λ' and β'). Suppose that λ' and β' are the same. Then any (odd) term λ'_i occurs the same number of times (say m) both in λ' and β'. We can write m as $m = \sum_j 2^{t_j}$ and $m = \sum_j 2^{s_j}$, where the sum runs over those js for

which $\lambda_j = \lambda_i' 2^{t_j}$ (and similarly for β_j). Since all λ_j (β_j) are distinct it follows that the sets of t_js and s_js are equal as well. Thus $\lambda = \beta$.
surjectivity: For each number-partition of n into odd terms λ_i's, we can count how often each of these odd terms appears. For example, assume that λ_i' appears in λ' exactly m times. We can consider the (unique) binary expansion of $m = \sum_j 2^{t_j}$ with t_js being all distinct. Now replacing all terms λ_i' in λ' by the terms of the form $\lambda_i' \cdot 2^{t_j}$ clearly yields the desired number-partition into distinct terms.

Problem 4 []

This is the case when we put distinguishable balls into indistinguishable boxes.

(a) 0 if $n > k$ and 1 if $n \leq k$ (injectivity - every box contains at most one ball)

(b) $S_{n,k}$ (surjectivity - every box contains at least one ball)

(c) $\sum_{j=0}^{k} S_{n,j}$ (there are no restrictions)

Problem 5 []

Consider a set $Z = X \dot{\cup} Y$ of $m+n$ points, where $X = \{x_1, \ldots, x_n\}$ is an n-set of blue points and Y is an m-set of red points. How many k-subsets consist of red points only? The answer is obviously $\binom{m}{k}$. Let A_i be those k-subsets that contain x_i. Then $|\cap_{i \in I} A_i| = \binom{m+n-|I|}{k-|I|}$. Since there are $\binom{m+n}{k}$ k-subsets of Z, by the principle of the inclusion and exclusion we obtain that the number of k-subsets that do not contain any points from X is:

$$\binom{m+n}{k} - \sum_{\emptyset \neq I \subseteq X} (-1)^{|I|-1} |\cap_{i \in I} A_i| =$$

$$\binom{m+n}{k} - \sum_{\emptyset \neq I \subseteq X} (-1)^{|I|-1} \binom{m+n-|I|}{k-|I|} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{m+n-i}{k-i}$$

The desired identity follows.