Solutions for the Example sheet 4

Problem 1 []

(a) Out of three numbers at least two have the same sign, thus, their product is non-negative.

(b) Every guest knows between 0 and $n-1$ people (pigeons). In total there are n options (holes). But the situation when one of the guests does not know anyone and some other guest knows everybody is mutually exclusive. Therefore, in the both scenarios above we actually have n people and $n-1$ options. By the simple form of the pigeonhole principle there are two persons who know the same number of guests.

(c) By factoring out as many as 2's as possible, we see that any integer can be written in the form $2^k \cdot a$, where $k \in \mathbb{N}_0$ and a is odd. For an integer between 1 and 200, a is one of the 100 numbers 1, 3, ..., 199. Thus among the 101 integers there are two having a's of equal value when written in this form. Let these two numbers be $2^r \cdot a$ and $2^s \cdot a$. If $r < s$, then the second number is divisible by the first. Otherwise, the first number is divisible by the second.

The same conclusion doesn't hold for just 100 numbers, as easily seen by taking: 101,..., 200.

Problem 2 []

For $n \in \mathbb{N}$ let $\mathcal{P}_n := \{x^n + \sum_{i=0}^{n-1} a_i x^i \mid a_i \in \mathbb{F}_p\}$ be the set of all polynomials of degree n with the leading coefficient 1 (monic polynomials of degree n). It holds $|\mathcal{P}_n| = p^n$, since we have exaclty p choices for each of the coefficients $a_0, \ldots, a_{n-1}$. For given $a \in \mathbb{F}_p$ we define the following sets $P_a := \{f \in \mathcal{P}_n \mid f(a) = 0\}$. Further let N_n be the set of all polynomials from $\mathcal{P}_n$, which do not take on the value 0. We see immediately:

$$N_n = \mathcal{P}_n \setminus \cup_{a\in\mathbb{F}_p} P_a.$$

The inclusion-exclusion principle yields:

$$|N_n| = |\mathcal{P}_n| - \sum_{\emptyset \neq A \subseteq \mathbb{F}_a} (-1)^{|A|-1} |\cap_{a\in A} P_a|. \tag{1}$$

To simplify (1) further, we notice:

$$P_a = (x-a) \cdot \mathcal{P}_{n-1} \quad \text{und} \quad |P_a| = p^{n-1}.$$

Moreover, for $A \subseteq \mathbb{F}_p$, $|A| \le n$ we have:

$$\bigcap_{a\in A} P_a = \left(\prod_{a\in A} (x-a) \right) \mathcal{P}_{n-|A|} \quad \text{und} \quad \left| \bigcap_{a\in A} P_a \right| = p^{n-|A|},$$

and for $|A| > n$ we obtain $|\cap_{a\in A} P_a| = 0$ (a polynomial of degree n has at most n zeros). Finally we simplify (1) as follows:

$$|N_n| = \sum_{A\subseteq \mathbb{F}_p} (-1)^{|A|} |\cap_{a\in A} P_a| = \sum_{k=0}^{p} (-1)^k \sum_{A\subseteq \mathbb{F}_p, |A|=k} |\cap_{a\in A} P_a| =$$
$$\sum_{k=0}^{\min\{p,n\}} (-1)^k \binom{p}{k} p^{n-k} = \sum_{k=0}^{n} (-1)^k \binom{p}{k} p^{n-k}.$$

Problem 3 []

The number of *red-blue* colorings of $[2n]$ with the property that if i is red then $i-1$ is not blue is exactly $2n+1$. Indeed, if i is red then $i-1$ is red, then $i-2$ is red and so on. Thus, it suffices to count the coloring which color first $i \in \{0\} \cup [2n]$ red. Next we interpret the expression from the exercise as the inclusion-exclusion formula applied to the problem above.

We consider $\mathbf{a} \in \{\text{red}, \text{blue}\}^{2n}$ which correspond exactly to the the red-blue-colorings of $[2n]$ (color i by a_i). We define the following sets of red-blue-sequences of the length $2n$. For $i \in [2, 2n] := \{2, 3, \ldots, 2n\}$ let

$$A_i := \left\{ \mathbf{a} \in \{\text{red}, \text{blue}\}^{2n} \mid a_i = \text{red}, a_{i-1} = \text{blue} \right\}.$$

We denote the set of those sequences by A, where all positions are blue or the first k positions are red and the other positions $[2n] \setminus [k]$ are blue. As already explained above: $|A| = 2n + 1$. Moreover:

$$A = \{\text{red}, \text{blue}\}^{2n} \setminus \cup_{i\in[2,2n]} A_i.$$

And by the IE principle:

$$|A| = 2^n - \sum_{\emptyset \neq I \subseteq [2,2n]} (-1)^{|I|-1} |\cap_{i\in I} A_i|.$$

Let's look closer at the terms $|\cap_{i\in I} A_i|$. By the definition of A_is we have $|\cap_{i\in I} A_i| = 0$ if I contains two consecutive numbers. From Problem 3 (Example sheet 2) we know that there are $\binom{2n-k}{k}$ many such k-element subsets , which contain no two consecutive numbers (especially for $|I| > n$ we have $|\cap_{i\in I} A_i| = 0$). For such subsets it holds:

$$|\cap_{i\in I} A_i| = 2^{2n-2i}.$$

Thus, we further rewrite the inclusion-exclusion formula and obtain:

$$2n + 1 = |A| = 2^{2n} - \sum_{\emptyset \neq I \subseteq [2,2n]} (-1)^{|I|-1} |\cap_{i\in I} A_i| = \sum_{i=0}^{n} (-1)^i \binom{2n-i}{i} 2^{2n-2i}.$$

Problem 4 []

Let $(d_i)_{i\in[30]}$ be the number of candies which the child has eaten from the first till (inclusively) the ith of April. It is clear that $d_i \neq d_j$ for $i \neq j$, since she ate every day at least one candy. Next we define the second sequence of the numbers $(e_i)_{i\in[30]} = (d_i + 14)_{i\in[30]}$. Also, for the e_is we have: $i \neq j \Rightarrow e_i \neq e_j$. In total we have 60 numbers (30 d_is und 30 e_is), which are contained in [59] (the child will have eaten till the 30th of April at most 45 candies and $45 + 14 = 59$). Therefore, there are two numbers which are equal (pigeonhole principle). Since these two numbers are neither both from $\{d_i : i \in [30]\}$ nor from $\{e_i : i \in [30]\}$, it follows that that there are i and j such that $d_i = e_j$. This means that $d_i = d_j + 14$, and therefore the child ate from the $(j + 1)$th April till ith of April exactly 14 candies.

Problem 5 []

Recall that $\varphi(n)$ was defined as the number of $m \in [n]$ which are relatively prime to n.
Let S denote the set of the pairs (f, d) of positive integers that satisfy

$$d|n \quad \text{and} \quad 1 \leq f \leq d \quad \text{and} \quad \gcd(f, d) = 1.$$

Then, obviuosly:

$$|S| = \sum_{d|n,\ d\in\mathbb{N}} |\{f\colon f \in [d], \gcd(f, d)\}| = \sum_{d|n,\ d\in\mathbb{N}} \varphi(d).$$

It is sufficient therefore to prove: $|S| = n$.
Indeed, define

$$g : \begin{cases} S \to [n] \\ (f, d) \mapsto f \cdot n/d. \end{cases}$$

Notice that since d divides n and $f \in [d]$, $g(f, d)$ is an integer in $[n]$.
injectivity of g:

$$g(f, d) = g(f', d') \Rightarrow fn/d = f'n/d' \Rightarrow fd' = f'd,$$

and, since f and d are coprime, $f = f'$ and $d = d'$.
surjectivity of g: Given $x \in [n]$, define $y := \gcd(x, n)$. Then set $d_x = n/y$ and $f_x = x/y$, which implies: $\gcd(f_x, d_x) = 1$. Thus, $(f_x, d_x) \in S$. Observe now:

$$g(f_x, d_x) = f_x n/d_x = \frac{xn}{y} \cdot \frac{y}{n} = x,$$

implying surjectivity of g.
Since g is bijective, we obtain by the Rule of Bijection: $|S| = n$ and the desired claim follows.