Solutions for the Example sheet 5

**Problem 1** []

First we show the *asymmetric* version of the theorem of Erdős and Szekeres from the lecture:

**Theorem 1.** *Every sequence* $(a_i)_{i\in[rs+1]}$ *of* $rs+1$ *real numbers contains either an increasing subsequence of length* $s+1$ *or a decreasing subsequence of length* $r+1$.

*Proof.* Almost verbatim as in the lecture. Indeed, Suppose there is no increasing subsequence of length $s+1$. Then we define for each $k \in [sr+1]$ an integer $m_k$ to be the length of the longest increasing subsequence which begins with $a_k$. Clearly, for every $k \in [sr+1]$ we have $m_k \in [s]$ and therefore, by the pigeonhole principle, there are indices $i_1 < \ldots < i_{r+1}$ such that $m_{i_1} = \ldots = m_{i_{r+1}}$. Further, by the definition of $m_k$s it is easily seen that $a_{i_1} \geq a_{i_2} \geq \ldots \geq a_{i_{r+1}}$. □

Let $(b_i)_{i\in[n]}$ be an arbitrary sequence of reals with $n = srp+1$. We show that either there is a strictly increasing subsequence of length greater than $s$, a strictly decreasing subsequence of length greater than $r$, or a constant subsequence of length greater than $p$. Suppose that among the numbers $b_1, \ldots, b_n$ there occur at most $sr$ distinct values. Then by the strong form of the pigeonhole principle there is a number that occurs at least $p+1$ times.
Assume now that none of the numbers occurs more than $p$ times. But then, again by the pigeonhole principle, there must exist among $b_i$s at least $sr+1$ distinct numbers. Therefore, the asymmetric version of the theorem of Erdős and Szekeres yields that, in this case, there is a strictly increasing subsequence of length greater than $s$ or a strictly decreasing subsequence of length greater than $r$.

**Problem 2** []

(a) Consider the function $f(t) = e^t - 1 - t$. Its derivative is $f'(t) = e^t - 1$ and the second derivative is $f''(t) = e^t > 0$. Therefore $f'(t)$ is strictly increasing on $\mathbb{R}$. Moreover, $f'(t) = 0$ iff $t = 0$. And for $t = 0$ we have $f(t) = 0$. Since $f'(t) > 0$ for $t > 0$ (and $f'(t) < 0$ for $t < 0$) we see that at 0 $f$ achieves its global minimum. And therefore $e^t > t+1$ for all $t \neq 0$.

(b) Let $k, n \in \mathbb{N}$. With (a) we obtain:

$$\left(1+\frac{k}{n}\right)^n < (e^{k/n})^n = e^k.$$

On the other hand, binomial theorem yields:

$$\left(1+\frac{k}{n}\right)^n > \binom{n}{k}\left(\frac{k}{n}\right)^k,$$

and therefore:

$$\binom{n}{k} < \left(\frac{k}{n}\right)^{-k} e^k = \left(\frac{ne}{k}\right)^k.$$

**Problem 3** []

(a) $R(k,k,k)$ is the smallest integer $n$ such that no matter how one colors the edges of $K_n$ with three colors, there is a monochromatic copy of $K_k$ in it. Formally:

$$R(k,k,k) := \min\left\{ n \,\middle|\, \forall \chi\colon \binom{[n]}{2} \to [3],\ \exists S \in \binom{[n]}{k} \text{ s.t. } \left|\chi\left(\binom{S}{2}\right)\right| = 1 \right\}.$$

(b) Let $\mathcal{C} := \{\chi \,|\, \chi : E(K_n) \to \{\text{green, blue, red}\}\}$ be the set of all edge-colorings of $K_n$ with three colors. Clearly: $|\mathcal{C}| = 3^{\binom{n}{2}}$. Further set $\mathcal{C}_S := \{\chi \in \mathcal{C} \,|\, |\chi(\binom{S}{2})| = 1\}$ to be the set of those edge-colorings of $K_n$ that contain a monochromatic $K_k$ on the vertex set $S \in \binom{[n]}{k}$. Observe: $|\mathcal{C}_S| = 3^{\binom{n}{2}-\binom{k}{2}+1}$, since we have $\binom{n}{2}-\binom{k}{2}$ edges in $K_n$ not contained in $S$ (which can be colored arbitrary) and we have three choices for the color of the edges from $\binom{S}{2}$. We consider

$$\left|\cup_{S\in\binom{[n]}{k}} \mathcal{C}_S\right| \le \sum_{S\in\binom{[n]}{k}} |\mathcal{C}_S| = \binom{n}{k} \cdot 3^{\binom{n}{2}-\binom{k}{2}+1}.$$

If the following holds

$$\binom{n}{k} \cdot 3^{\binom{n}{2}-\binom{k}{2}+1} < 3^{\binom{n}{2}} = |\mathcal{C}| \tag{1}$$

then $\mathcal{C} \setminus \cup_{S\in\binom{[n]}{k}} \mathcal{C}_S \neq \emptyset$. (This means: $R(k,k,k) > n$. )

The inequality (1) is equivalent to

$$\binom{n}{k} 3^{-\binom{k}{2}+1} < 1 \tag{2}$$

Furthermore:

$$\binom{n}{k} 3^{-\binom{k}{2}+1} \quad < \quad \left(\frac{ne}{k}\right)^k 3^{-\binom{k}{2}+1} \quad < \quad \left(\frac{ne}{k}\right)^k 3^{-\frac{k^2}{2}+k} \quad = \quad \left(\frac{3en}{k\sqrt{3}^k}\right)^k,$$

implying for $n \le \frac{k}{3e}\sqrt{3}^k$ the inequality (2) (and (1)). Therefore there is an edge-coloring of $K_n$ without a monochromatic $K_k$, i.e. $R(k,k,k) > \frac{k}{3e}\sqrt{3}^k$.

**Problem 4** []

(a) We define for $r \in \mathbb{N}$ the Ramsey number as follows:

$$R_r(3) = \min\left\{ n \in \mathbb{N} \,\middle|\, \forall \chi\colon \binom{[n]}{2} \to [r]\, \exists\, S \subset [n], |S| = 3 : |\chi\left(\binom{S}{2}\right)| = 1 \right\}.$$

In words: $R_r(3)$ is the smallest integer $n$ such that in every coloring of the edges of $K_n$ with $r$ colors there is a monochromatic triangle ($K_3$).

(b) First we prove the following recursive formula for the Ramsey numbers $R_r(3)$:

$$R_r(3) \leq r(R_{r-1}(3) - 1) + 2. \tag{3}$$

Suppose the contrary, i.e. that $R_r(3) > r(R_{r-1}(3) - 1) + 2$. Then there is an edge-coloring of $K_n$ with $n = r(R_{r-1}(3) - 1) + 2$ without monochromatic triangles. Pick an arbitrary vertex, say $n$. It has $n - 1 = r(R_{r-1}(3) - 1) + 1$ neighbours, and by the strong form of the pigeonhole principle, there is a color $i \in [r]$ such that $n$ is incident to at least $R_{r-1}(3)$ edges colored $i$. Call $V$ the set of vertices $n$ is adjacent to by color $i$. By the above argument: $|V| \geq R_{r-1}(3)$. None of the edges from $\binom{V}{2}$ can be colored with the color $i$, thus $\binom{V}{2}$ is colored with at most $r - 1$ colors. Since $|V| \geq R_{r-1}(3)$, there is a monochromatic triangle in $V$, and therefore in $K_n$, a contradiction.

We can show that $R_r(3)$ is finite for all $r \in \mathbb{N}$ by induction, using (3). Clearly: $R_1(3) = 3$ (induction start). Induction hypothesis $\mathcal{A}(r) : R_r(3) < \infty$. Induction step: From $R_{r+1} \leq (r+1)(R_r(3) - 1) + 2$ and $R_r < \infty$ it clearly follows that $R_{r+1}(3)$ is finite.

(c) Again we use induction on $r$ and (3).
Induction start:
For $r = 1$ we have $R_1(3) = 3 = 2 + 1 = \lfloor e \cdot 1! \rfloor + 1$.
Induction hypothesis $\mathcal{A}(r)$: gilt $R_r(3) \leq \lfloor e \cdot r! \rfloor + 1$.
$\mathcal{A}(r) \Longrightarrow \mathcal{A}(r+1)$. Let $r \geq 1$.

$$\begin{aligned}
R_{r+1}(3) &\overset{(3)}{\leq} (r+1) \cdot (R_r(3) - 1) + 2 \overset{\mathcal{A}(r)}{\leq} (r+1)(\lfloor er! \rfloor + 1 - 1) + 2 \\
&= (r+1) \left\lfloor \sum_{i=0}^{\infty} \frac{r!}{i!} \right\rfloor + 2 = (r+1) \left\lfloor \sum_{i=0}^{r} \frac{r!}{i!} + \sum_{i=r+1}^{\infty} \frac{r!}{i!} \right\rfloor + 2 \\
&= (r+1) \sum_{i=0}^{r} \frac{r!}{i!} + \frac{(r+1)!}{(r+1)!} + 1 + (r+1) \left\lfloor \sum_{i=0}^{\infty} \frac{r!}{(r+1+i)!} \right\rfloor \\
&= \sum_{i=0}^{r+1} \frac{(r+1)!}{i!} + 1 + (r+1) \left\lfloor \sum_{i=0}^{\infty} \frac{r!}{(r+1+i)!} \right\rfloor
\end{aligned}$$

Next we show that $\left\lfloor \sum_{i=0}^{\infty} \frac{r!}{(r+1+i)!} \right\rfloor = 0$. It suffices to prove that the above series in $\lfloor \cdot \rfloor$ is less than 1:

$$\sum_{i=0}^{\infty} \frac{r!}{(r+1+i)!} = \sum_{i=0}^{\infty} \frac{1}{(r+1+i) \cdot \ldots \cdot (r+1)} < \frac{1}{r+1} \sum_{i=0}^{\infty} \frac{1}{(r+1)^i} = \frac{1}{r} \leq 1.$$

Therefore we obtain from the estimates above:

$$R_{r+1}(3) \leq e \cdot (r+1)! + 1.$$

Since $R_{r+1}(3) \in \mathbb{N}$ we can round down to get:

$$R_{r+1}(3) \leq \lfloor e \cdot (r+1)! \rfloor + 1.$$

**Problem 5** []

We take $k-1$ many disjoint sets $V_1, \ldots, V_{k-1}$ each of cardinality $k-1$. We color the edges within each of the $S_i$s red and between any $S_i$ and $S_j$ $(i \neq j)$ blue (see the picture). By the pigeonhole principle, neither red nor blue $K_k$ exists (since among any $k$ vertices there are at least two that both lie in some $V_i$ and there are at least two that lie in different $V_i$s ) and the graph constructed has $(k-1)^2$ vertices.