SOLUTIONS FOR THE EXAMPLE SHEET 6

Problem 1 []

Consider the following product of power series:

$$\left(\sum_{i=0}^{\infty} x^i\right)\left(\sum_{j=0}^{\infty} x^{2j}\right)\left(\sum_{k=0}^{\infty} x^{5k}\right).$$

The coefficient of x^n is exactly the number of ways to write n as the following sum

$$i + 2j + 5k,$$

where $i, j, k \in \mathbb{N}_0$. Clearly, this corresponds to the number of ways to pay n dollars using 1 and 2 dollar coins and 5 dollar bills if we interpret i as the number of 1 dollar coins, j as the number of 2 dollar coins and k as the number of 5 dollar bills. Moreover, we know from the lecture:

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x},$$

replacing x in the formula above by x^2, x^5, respectively, we obtain:

$$\sum_{i=0}^{\infty} x^{2i} = \frac{1}{1-x^2} \quad \text{und} \quad \sum_{i=0}^{\infty} x^{5i} = \frac{1}{1-x^5}.$$

and the generating function is:

$$\frac{1}{(1-x)(1-x^2)(1-x^5)}.$$

Problem 2 []

(a) Let the number of allowed sequences of length n be a_n. We classify the sequences according to their first symbols. There are a_{n+2} $\{0,1\}$-sequences of length $n+3$ that start with 1. There are a_{n+1} $\{0,1\}$-sequences of length $n+3$ that start with 01. There are a_n $\{0,1\}$-sequences of length $n+3$ that start with 001. The classifications above are mutually exclusive and exhaust all possibilites. Thus, we obtain the following linear recursion:

$$a_{n+3} = a_{n+2} + a_{n+1} + a_n \quad (\forall n \geq 0). \tag{1}$$

The initial conditions are clearly $a_0 = 1$, $a_1 = 2$ and $a_2 = 4$.

(b) First we write down the characteristic polynomial $p(x) = x^3 - x^2 - 9x + 9 = (x-1)(x+3)(x-3)$ and determine its roots to be -3, 1 and 3. By a theorem from the lecture we obtain the following general solution of the recursion:

$$a_n = c_1 \cdot (-3)^n + c_2 \cdot 1^n + c_3 \cdot 3^n. \tag{2}$$

We know the initial values $a_0 = 0$, $a_1 = 1$ and $a_2 = 2$. Therefore we substitute $n = 0$, $n = 1$ and $n = 2$ into (2). Solving the system of linear equalities:

$$\begin{cases} c_1 + c_2 + c_3 = 0 \\ -3c_1 + c_2 + 3c_3 = 1 \\ 9c_1 + c_2 + 9c_3 = 2. \end{cases}$$

we obtain

$$c_1 = -\frac{1}{12} \quad \text{und} \quad c_2 = -\frac{1}{4} \quad \text{und} \quad c_3 = \frac{1}{3}.$$

The formula is therefore:

$$a_n = -\frac{1}{12}(-3)^n - \frac{1}{4} + \frac{1}{3}3^n.$$

Problem 3 []

We claim that the following identities hold:

(a) $F_1 + F_3 + \ldots + F_{2n-1} = F_{2n}$ and

(b) $F_0 + F_2 + \ldots + F_{2n} = F_{2n+1} - 1$.

(a)

$$\mathcal{A}(n): \quad F_1 + F_3 + \ldots + F_{2n-1} = F_{2n}$$

Induction start: For $n = 1$ we have $F_1 = 1 = F_2$.
$\mathcal{A}(\mathbf{n}) \Longrightarrow \mathcal{A}(\mathbf{n+1})$:

$$F_1 + F_3 + \ldots + F_{2n-1} + F_{2(n+1)-1} \overset{\mathcal{A}(n)}{=} F_{2n} + F_{2(n+1)-1} = F_{2(n+1)}.$$

(b)

$$\mathcal{B}(n): \quad F_0 + F_2 + \ldots + F_{2n} = F_{2n+1} - 1$$

Induction start: For $n = 0$ we have $F_0 = 0 = F_1 - 1$.
$\mathcal{B}(\mathbf{n}) \Longrightarrow \mathcal{B}(\mathbf{n+1})$:

$$F_0 + F_2 + \ldots + F_{2n} + F_{2(n+1)} \overset{\mathcal{B}(n)}{=} (F_{2n+1} - 1) + F_{2(n+1)} = F_{2(n+1)} - 1.$$

Problem 4 []

(a) Let y_n be the number of ways to lay out a path of n tiles. We classify after the last nth tile. If its color is green or gray, so there is no restriction for the $(n-1)$st tile. Thus there are y_{n-1} ways to lay out the further way. If the color of the nth tile is red, then the $(n-1)$st tile can only be either green or gray, and afterwards, we can tile the remaining path in y_{n-2} ways. Thus, we obtain the following recursion:

$$y_n = 2y_{n-2} + 2y_{n-1}, \tag{3}$$

where 2 "stands" for the colors green or gray (in both considerations).

(b) The initial conditions are $y_1 = 3$ and $y_2 = 8$ (no two red tiles side by side).

(c) The characteristic polynomial is $p(x) = x^2 - 2x - 2$ with the roots $\lambda_1 = 1+\sqrt{3}$ and $\lambda_2 = 1 - \sqrt{3}$. The general formula for the recursion (3) is

$$y_n = c_1(1+\sqrt{3})^n + c_2(1-\sqrt{3})^n. \tag{4}$$

For the easier computation of the coefficients c_1 and c_2 we introduce $y_0 = 1$. It is easily seen that the recursion (3) holds for $n \geq 0$. Similarly as in the second problem we solve the system of linear equations:

$$\begin{cases} c_1 + c_2 = 1 \\ c_1(1+\sqrt{3}) + c_2(1-\sqrt{3}) = 3. \end{cases}$$

We obtain the following coefficients

$$c_1 = \frac{2+\sqrt{3}}{2\sqrt{3}} \quad \text{und} \quad c_2 = \frac{\sqrt{3}-2}{2\sqrt{3}}.$$

Therefore, the recursion for the initial conditions in (b) is as follows:

$$y_n = \frac{2+\sqrt{3}}{2\sqrt{3}}(1+\sqrt{3})^n + \frac{\sqrt{3}-2}{2\sqrt{3}}(1-\sqrt{3})^n.$$

And $y_{15} = 3799168$.

Problem 5 []

Let $A(x)$ be the generating function for the sequence (a_n). From the recurrence relation

$$a_{n+2} = 3a_{n+1} + 4a_n$$

it follows that

$$\frac{1}{x^2}\sum_{n=0}^{\infty} a_{n+2}x^{n+2} = \frac{3}{x}\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + 4\sum_{n=0}^{\infty} a_n x^n,$$

where we set $a_0 = 0$, since $a_2 = 3a_1 + 4a_0$. That is,

$$\frac{1}{x^2}(A(x) - x - 3x^2) = \frac{3}{x}(A(x) - x) + 4A(x),$$

from which it follows that

$$A(x) = \frac{x}{1-3x-4x^2} = \frac{1}{(1+x)(1-4x)} = \frac{1}{5}\left(\frac{1}{1-4x} - \frac{1}{1+x}\right) = \frac{1}{5}\left(\sum_{n=0}^{\infty} 4^n x^n - \sum_{n=0}^{\infty}(-1)^n x^n\right),$$

where the penultimate equality follows from partial fractions decomposition, and the last equality follows from generalized binomial theorem. If we equate the coefficients now, we obtain:

$$a_n = \frac{1}{5}(4^n - (-1)^n).$$