Solutions for the Example sheet 7

Problem 1 []

(a) The generating function of $(1, 0, 1, 0, 1, 0, \ldots)$ is $\frac{1}{1-x^2}$ (see (b)), and therefore the generating functions of $(-6, 0, -6, 0, -6, 0, \ldots)$ and $(6, 0, 6, 0, 6, 0, \ldots)$ are $\frac{-6}{1-x^2}$ and $\frac{6}{1-x^2}$ respectively (multiply by $-6/6$), now we shift the elements of the former sequence by 4 positions to the right (multiply by x^4) and the elements of $(6, 0, 6, 0, 6, \ldots)$ by 5 positions to the right, multiplication by x^5. Addition yields:

$$\frac{6x^5 - 6x^4}{1 - x^2},$$

and after simplifying:

$$\frac{-6x^4}{1 + x}.$$

(b) The generating function of the sequence $(1, 0, 1, 0, 1, 0, \ldots)$ is obtained by substituting x^2 into the generating function of $(1, 1, 1, 1, 1, \ldots)$ (which is $\frac{1}{1-x}$). Thus, we obtain $\frac{1}{1-x^2}$.

(c) To obtain the generating function of $(1, 2, 1, 4, 1, 8, \ldots)$, we add the generating functions of $(1, 0, 1, 0, 1, 0, \ldots)$ and $(0, 2, 0, 4, 0, 8, \ldots)$. The latter is $\frac{2x}{1-2x^2}$, since $\frac{1}{1-2x^2}$ corresponds to $(1, 0, 2, 0, 4, 0, 8, \ldots)$ (insert $2x^2$ for x into $\frac{1}{1-x}$), and then shift by multiplying by $2x$). Thus, the generating function is

$$\frac{1}{1 - x^2} + \frac{2x}{1 - 2x^2} = \frac{1 + 2x - 2x^2 - 2x^3}{(1 - 2x^2)(1 - x^2)}.$$

(d) The generating function of $(0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots)$ is $\frac{x^2}{1-x^3}$, and thus, the generating function of $(1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots)$ is

$$\frac{1}{1 - x} - \frac{x^2}{1 - x^3} = \frac{1 + x}{1 - x^3}.$$

Problem 2 []

(a) The generating function of (a_n), where $a_n = 1$ (for all n) is $A(x) = \frac{1}{1-x}$. Differentiating twice we obtain:

$$A''(x) = \sum_{n=0}^{\infty} n(n-1)x^{n-2} = \frac{d^2}{dx^2}\left(\frac{1}{(1-x)}\right) = \frac{2}{(1-x)^3}.$$

Shift the sequence by mulitplying by x^2 and obtain the desired generating function to be

$$A_a(x) = \frac{2x^2}{(1-x)^3}.$$

(b) The generating function of (a_n), where $a_n = 1$ (for all n) is $A(x) = \frac{1}{1-x}$. Differentiating $A(x)$ we obtain $\frac{1}{(1-x)^2}$ which is the generating function for the sequence (a_n) with $a_n = n+1$. Multiplying by x we obtain the sequence $(0, 1, 2, 3, \ldots)$ with the generating function $B(x) = \frac{x}{(1-x)^2}$. Now differentiate $B(x)$ obtaining

$$B'(x) = \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \frac{1+x}{(1-x)^3} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} nx^n\right) = \sum_{n=0}^{\infty} n^2 x^{n-1}.$$

And again, multiplication by x gives the generating function for $(a_n = n^2)$:

$$A_b(x) = \frac{x + x^2}{(1-x)^3}.$$

(c) Differentiating the function $A_b(x)$ and multiplying by x yields the generating function for the cubes, i.e. (n^3):

$$x\frac{(1+2x)(1-x) + 3(x+x^2)}{(1-x)^4} = \frac{x(1+4x+x^2)}{(1-x)^4}.$$

Problem 3 []

Let h_n denote the number of allowed sequences. The allowed sequences have the form aS_1 or $01S_2$, where a is 1 or 2 and S_1, S_2 are sequences of lengths $n+1$, n respectively, with the required property. It follows that

$$h_{n+2} = 2h_{n+1} + h_n \quad (\forall n \geq 0).$$

Also $h_0 = 1$ and $h_1 = 3$. Multiplying both sides by x^n and summing over all $n \in \mathbb{N}_0$ we have:

$$\sum_{n=0}^{\infty} h_{n+2}x^n = \sum_{n=0}^{\infty} 2h_{n+1}x^n + \sum_{n=0}^{\infty} h_n x^n,$$

which gives the following relation involving the generating function $H(x)$ of (h_n):

$$\frac{1}{x^2}(H(x) - 1 - 3x) = \frac{2}{x}(H(x) - 1) + H(x).$$

And it follows that

$$H(x) = \frac{1+x}{1-2x-x^2}.$$

Problem 4 []

We mutliply both sides of $a_{n+1} = 2a_n + n$ by x^n and sum over all $n \in \mathbb{N}_0$:

$$\sum_{n=0}^{\infty} a_{n+1}x^n = \sum_{n=0}^{\infty} 2a_n x^n + \sum_{n=0}^{\infty} nx^n.$$

Notice that $\sum_{n=0}^{\infty} nx^n = x\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}$. Therefore we obtain:

$$\frac{1}{x}(A(x) - a_0) = 2A(x) + \frac{x}{(1-x)^2}.$$

Thus, obtaining:

$$A(x) = \frac{1 - 2x + 2x^2}{(1-x)^2(1-2x)}.$$

Partial fractions decomposition gives:

$$A(x) = \frac{-1}{(1-x)^2} + \frac{2}{1-2x}.$$

By the binomial theorem,

$$A(x) = -\sum_{n=0}^{\infty} \binom{-2}{n}(-x)^n + 2\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty}(2^{n+1} - n - 1)x^n,$$

and therefore, the sequence is $a_n = 2^{n+1} - n - 1$.

Problem 5 []

(a) We have to break at each of $(n-1)$ points; the procedure is determined if we specify the order in which we break at the different places. Hence, the number is $(n-1)!$.

(b) We define h_n to be the number of ways to break a stick of length n into n pieces of unit length. Set $h_0 = 0$, and, clearly, $h_1 = h_2 = 1$. First we set up a recurrence relation involving different h_ns. If we have a stick of length n, then there are $n-1$ ways to start the first "move". Thus, if we first break a stick of length n into one of length k and the other of length $n-k$, then there are $h_k \cdot h_{n-k}$ ways to break these sticks further down to unit lengths. Therefore, we obtain the following nonlinear recurrence relation:

$$h_n = \sum_{k=1}^{n-1} h_k h_{n-k}. \tag{1}$$

But this is the recurrence relation from the lecture, and therefore, we know

$$h_n = \frac{1}{n-1}\binom{2n-2}{n-1}. \quad \text{(Catalan number)}$$

A very short solution (see lecture for more details): We multiply both sides of (1) by x^n and sum up over all $n \geq 2$:

$$H(x) - x = \sum_{n=2}^{\infty} h_n x^n = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} h_k h_{n-k} x^n =$$
$$\sum_{n=2}^{\infty}\sum_{k=0}^{n} h_k x^k h_{n-k} x^{n-k} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} h_k x^k h_{n-k} x^{n-k} = (H(x))^2.$$

Solving for $H(x)$ we obtain $H(x) = \frac{1-\sqrt{1-4x}}{2}$, since the other solution gives 1 when evaluated at 0 (but it needs to be equal to h_0). Applying binomial theorem to $H(x) = \frac{1-\sqrt{1-4x}}{2}$ we obtain:

$$H(x) = \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n.$$