Solutions for the Example sheet 8

Problem 1 []

(a) Assume that $d \geq 2$, otherwise $Q_d = K_1$, average degree is 0, the number of edges is 0, circumference is 0 and the girth is ∞ and the diameter is 0.

The number of vertices in Q_d is 2^d, and since every vertex has degree d (change any of the d coordinates to obtain its neighbors), the average degree of Q_d is d. From this we get that Q_d has $d2^{d-1}$ edges (by the handshaking lemma). The girth of Q_d is 4 since it does not contain a cycle of length 3 (if $uvwu$ were a cycle of length 3 in Q_d then the number of 1s in u is equal to the number of 1s in w modulo 2, and therefore uw is not an edge, a contradiction).

The diameter of Q_d is d since: the shortest path between $\{0\}^d$ and $\{1\}^d$ has length d (we have to change d coordinates of $\{0\}^d$ to arrive at $\{1\}^d$); moreover, given $x, y \in V(Q_d)$, it is sufficient to change entries of the coordinates at which x and y differ (and these are at most d many).

The circumference of Q_d is 2^d. This is best shown by induction.
$\mathcal{P}(d)$**:** There is a cycle C of length 2^d which contains the edge $\{(0,0,\dots,0),(1,0,\dots,0)\}$.
For $d = 2$ this is obvious.
$\mathcal{P}(d-1) \Longrightarrow \mathcal{P}(d)$**:** Now, split the vertex set of Q_d into two parts: $\{0,1\}^{d-1} \times \{0\}$ and $\{0,1\}^{d-1} \times \{1\}$. Within each of the parts we find the cycles C and C' of length 2^{d-1} each, such that C contains the egde $\{(0,0,\dots,0),(1,0,\dots,0)\}$ and C' contains the edge $\{(0,0,\dots,1),(1,0,\dots,1)\}$. Let P be the path in C of length $2^{d-1} - 1$ with ends $(0,0,\dots,0)$ and $(1,0,\dots,0)$, and let P' be the path in C' of length $2^{d-1} - 1$ with ends $(0,0,\dots,1)$ and $(1,0,\dots,1)$. Since $\{(0,0,\dots,0),(0,0,\dots,1)\}$ and $\{(1,0,\dots,0),(1,0,\dots,1)\}$ are both edges in Q_d, these form together with P and P' a cycle of length 2^d in Q_d. All that remains is to observe that, by relabeling the positions 1 and d, we obtain a cycle of length 2^d in Q_d containing the edge $\{(0,0,\dots,0),(1,0,\dots,0)\}$.

(b) We define the following graph $\mathcal{G}$ on $\mathcal{P}([d])$. Its two vertices $A, B \in \mathcal{P}([d])$ are connected iff $|A\Delta B| = 1$ (symmetric difference). Further the isomorphism $\phi\colon V(\mathcal{G}) \to \{0,1\}^d$, by setting for every $A \in \mathcal{P}([d])$:

$$(\phi(A))_i = \begin{cases} 0, & \text{if } i \notin A, \\ 1, & \text{otherwise .} \end{cases}$$

Indeed, $|A\Delta B| = 1$ if, and oly if $\phi(A)$ and $\phi(B)$ differ in exactly one coordinate.

Problem 2 []

We may assume that $|C| < \sqrt{k}$, as otherwise there is nothing to be shown. And let $x, y \in V(C)$ be the two vertices and P be an x-y-path in G of length at least k. Assume further that $v_1, \dots v_\ell$ are those inner vertices of P that lie on C. Since $|C| < \sqrt{k}$, we obtain $\ell < \sqrt{k} - 2$. Furthermore, the path P is partitioned by its inner vertices into $\ell+1$ disjoint paths (no inner vertice in common), whose inner vertices are disjoint from $V(C)$. In particular (by the pigeonhole principle), one of these paths (call it P' and assume its ends are v_i and v_{i+1}) must have length $\frac{k}{\ell+1} \geq \frac{k}{\sqrt{k}-1} \geq \sqrt{k}+1$. Therefore, P' together with a path from v_i to v_{i+1} along the cycle C forms a cycle of length greater than $\sqrt{k}$.

Problem 3 []

(a) If G is disconnected then there is a bipartition of $V(G) = V_1 \dot\cup V_2$ such that $E(V_1, V_2) = \emptyset$. Therefore, the number of edges in G is at most $\binom{n}{2} - |V_1||V_2|$. Since we have that $|V_1||V_2| \geq n-1$ whenever $|V_1|+|V_2| = n$ and $|V_1|, |V_2| \geq 1$, it follows that $e(G) \leq \binom{n-1}{2}$, a contradiction to $e(G) > \binom{n-1}{2}$.

(b) Assume that G is disconnected and $V(G) = V_1 \dot\cup V_2$ such that $E(V_1, V_2) = \emptyset$ and $V_1, V_2 \neq \emptyset$. But this means that $\bar{G}$ contains all edges with one end in V_1 and the other in V_2, and thus contains K_{V_1,V_2}, which is clearly connected.

Problem 4 []

Define a graph G on the vertex set $S =: V(G)$ as follows: $xy \in E(G)$ iff $\| x-y \|_2 = 1$. If $e(G) > 3n$, there must be a vertex with degree at least 7. However, one cannot arrange on the cycle of radius 1 more than 6 points with pairwise distances at least 1 (Consider a unit disc around $(0,0)$. Indeed if there are more than 6 points on it with pairwise distances at least 1, this would imply that there are two points (vectors) with angle less than $\frac{2\pi}{6}$, implying that their distance is less than 1). A contradiction.

Problem 5 []

(a) draw pictures!

(b) follows from Problem 3 (b)

(c) Since G is self-complementary, we have $e(G) = e(\bar{G})$ and $e(G) + e(\bar{G}) = \binom{v(G)}{2}$. Therefore, $e(G) = \frac{v(G)(v(G)-1)}{4}$, implying $v(G) \equiv 0, 1 \pmod 4$.

(d) Let $\varphi\colon V(G) \to V(\bar{G})$ be an isomorphism from G to $\bar{G}$. If no vertex in G has degree $2k$, then let ℓ be the number of vertices of degree greater than $2k$ in G. But, this means that the number of vertices of degree greater than $2k$ in $\bar{G}$ is $4k+1-\ell$. Since φ is an isomorphism between G and $\bar{G}$, this means that $4k+1-\ell = \ell$, a contradiction since ℓ is an integer.