Solutions for the Example sheet 9

Problem 1 []

A graph G with at least $(|T|-1)|G|$ edges has average degree at least $2(|T|-1)$, and thus, by a proposition from the lecture, G contains a subgraph H with minimum degree at least $|T|-1$.
Next we show that $T \subseteq H$ by constructing an embedding $\varphi\colon V(T) \to V(H) \subseteq V(G)$. Set $t := |V(T)|$ and let $V(T) = \{x_1, \dots, x_t\}$, such that x_i has a unique neighbor in $\{x_1, \dots, x_{i-1}\}$ for all $i \in [t] \setminus \{1\}$ (another proposition from the lecture!). Pick an arbitrary vertex v_1 of $V(H)$ and set $\varphi(x_1) = v_1$. Assume that we already embedded $i-1$ many vertices, $i < t$, (i.e. we constructed an embedding from $T[\{x_1, \dots, x_{i-1}\}]$ into H). Let $j \le i-1$ and x_j be the unique neighbor of x_i. Then, since $\delta(H) \ge t-1$, there is a neighbor of $\varphi(x_j)$ in H (call it y), not yet used for an image of $\{x_1, \dots, x_{i-1}\}$. Setting $\varphi(x_i) := y$, we see that φ is now an embedding of $T[\{x_1, \dots, x_i\}]$ into H. At any step, a new vertex x_i can be embedded, and thus we manage to construct an embedding of T into H.

Problem 2 []

Let $P_\ell = x_0 x_1 \dots x_\ell$ be a longest path in G. If $\ell \ge 2\delta(G)$, then we are done. So, suppose that $\ell \le 2\delta(G) - 1$ and let x_0 and x_ℓ be the ends of P_ℓ.
If $x_0 x_\ell \in E(G)$, then $V(P_\ell) = V(G)$ since otherwise, by connectivity of G, we could obtain a longer path by appending a new edge to $P_\ell + x_0 x_\ell$ and deleting an appropriate edge on the cycle. Thus, $|G| = \ell + 1$ and we are done.
Assume now that $x_0 x_\ell \notin E(G)$. Further, $N(x_0)$, $N(x_\ell) \subseteq \{x_1, \dots, x_{\ell-1}\}$, since P_ℓ is a longest path. Let $x_{i_1}, \dots, x_{i_{\delta(G)}}$ be some neighbors of x_0 among $\{x_1, \dots, x_{\ell-1}\}$. Since $\ell - 1 \le 2(\delta(G) - 1)$, by the pigeonhole principle, there exists some i_j such that $x_{i_j - 1} \in N(x_\ell)$. But in this way, we obtain a cycle $C = x_\ell x_{i_j-1} x_{i_j-2} \dots x_0 x_{i_j} \dots x_\ell$ of length ℓ. Again, since G is connected and P_ℓ is a longest path, we get $\ell = |G| - 1$. Thus, if $\ell < 2\delta(G)$, then $\ell = |G| - 1$, and we have a path of length $\min\{2\delta(G), |G| - 1\}$.

Problem 3 []

Let T be a tree and φ an automorphism of it. Consider a longest path P in T. If P were unique, then clearly, such a longest path is mapped under φ onto itself. Therefore, if $|P|$ has $2\ell + 1$ vertices, the middle vertex is mapped onto itself, and is therefore a fixpoint. If $|P|$ is even, then the middle edge is mapped onto itself.
Suppose now that there are two distinct paths P' and P'' of maximum length. Next we claim that they share the middle vertex or the middle edge, based on the parity of $|P|$.
handy notation: If P is a path $x_0 \dots x_k$, then denote by Px_i the path $x_0 \dots x_i$ and by $x_i P$ the path $x_i \dots x_k$ for $i \in [k] \cup \{0\}$. By $G \cap G'$ we denote the graph $(V(G) \cap V(G'), E(G) \cap E(G'))$.
First observe that $V(P') \cap V(P'') \neq \emptyset$, since otherwise let P be a shortest path in T which connects one of the vertices of $V(P')$ with $V(P'')$. Let these vertices be x' and x''. Then assume that $P'x'$ is at least as long as $x'P'$ and assume that $x''P''$ is longer or equally long as $P''x''$. Then, the path $P'x'Px''P$ is longer than P', a contradiction.

Next we claim that $P' \cap P''$ is a path again. Indeed, otherwise let C_1 and C_2 be some connected components (paths again !) of $P' \cap P''$. But this is impossible since then the ends of the paths C_1 and C_2 are connected by different paths in T, a contradiction by Problem 5(2) (tree theorem).

Thus, $P' \cap P''$ is a path. Next we claim that it contains a middle edge of P' (and thus of P''), if $|P'|$ is even, and middle vertex if $|P'|$ is odd. If this is not the case, then we could find a longer path in T by proceeding as follows. Let $P := P'' \cap P''$, and its endvertices be a and b. Then, consider paths $P'a$, $P''a$, bP' and bP''. Since the middle edge (middle vertex) is not in P, we first assume that the middle edge (vertex) is in $P'a$ for P' and in $P''a$ for P''. But then, $P''a$ followed by the path $P'a$ is clearly longer than P' (and thus P''). Thus obtaining a contradiction in this case. In the case that the middle edge (vertex) is in $P'a$ for P' and is in bP'' for P'', we obtain the path $P'aPbP''$, which is longer than P', a contradiction as well. Other situations are symmetric, and thus we have that any two longest paths share their middle edge (vertex), which is clearly fixed under any automorphism.

Problem 4 []

This is very similar to the proof about Eulerian graphs from the lecture. Indeed, let

$$W = v_0 e_0 \dots e_{\ell-1} v_\ell$$

be a longest walk in G that traverses every edge of W exactly once in each direction. Since W cannot be extended, each of the edges at v_ℓ has been traversed twice (exactly once in each direction). Hence, $v_\ell = v_0$, and W is a closed walk. Suppose that W doesn't contain all the edges, then G has an edge E not traversed by W and incident with some vertex, say v_i. Let $e = vv_i$. But then the walk

$$W' = vev_ie_iv_{i+1} \dots e_{\ell-1}v_\ell e_0 \dots e_{i-1}v_iev$$

is longer and still has the required property, a contradiction.

Problem 5 []

This is Theorem 5.1.2 in J. Matoušek and J. Nešetřil, “Invitation to Discrete Mathematics”.