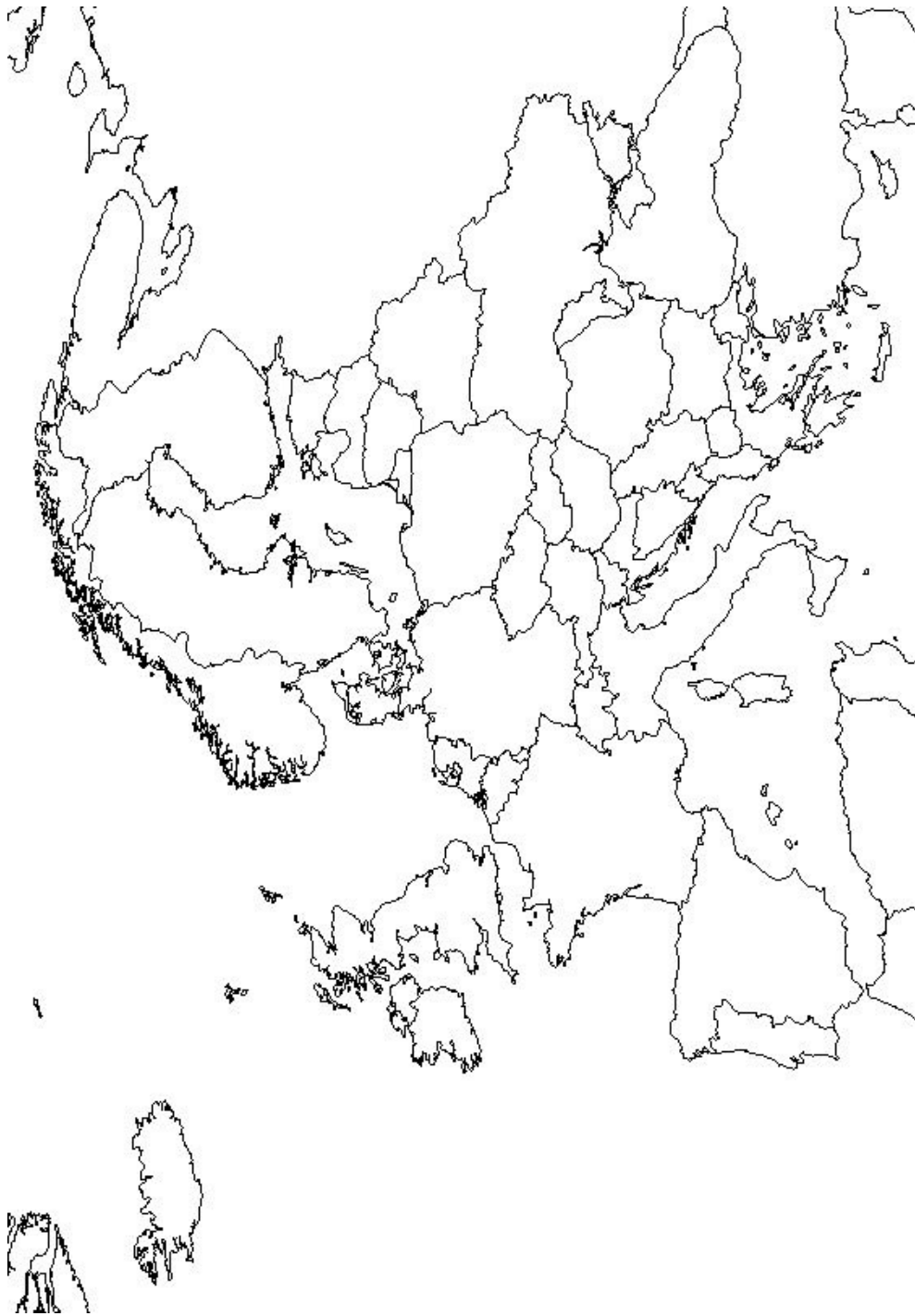
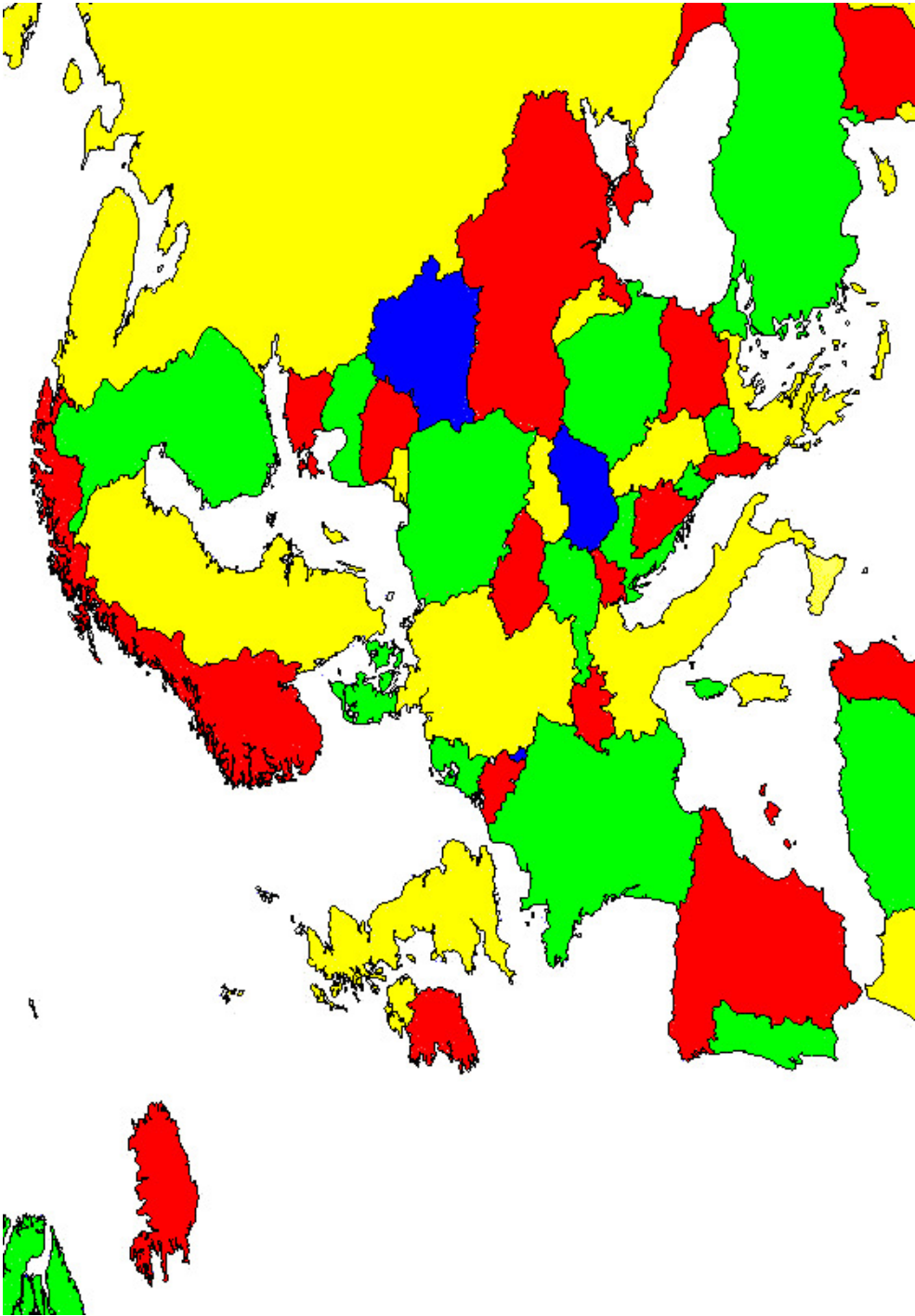


How many colors are needed to color a map?



Is 4 always enough? _____



Two relevant concepts

How many colors do we need to color a map so neighboring countries get different colors?

Simplifying assumption (not true in reality): Each country is bounded by a simple continuous curve.

Auxiliary graph: $V(G)$ = set of countries, $E(G)$ = pairs of countries that are neighboring (share a 1-dimensional piece of their boundary. (just points are not enough!))

Graph colorings: We then want a coloring of the *vertices* of this auxiliary graph, such that adjacent vertices receive distinct colors.

Planar graphs: The auxiliary graph G of the map has a special property: it can be drawn into the plane such that the edges do not cross. Indeed: draw the vertex representing the country in the “middle” (the “capitol”) and draw a curve to the middle of the boundary curve with each country. This drawing forms an embedding of the graph G in the plane so that the edges do not intersect.

Vertex coloring, chromatic number_____

A k -coloring of a graph G is a labeling $f : V(G) \rightarrow S$, where $|S| = k$. The labels are called colors; the vertices of one color form a color class.

A k -coloring is proper if adjacent vertices have different labels. A graph is k -colorable if it has a proper k -coloring.

The chromatic number is

$$\chi(G) := \min\{k : G \text{ is } k\text{-colorable}\}.$$

A graph G is k -chromatic if $\chi(G) = k$.

Examples. K_n , $K_{n,m}$, C_5 , Petersen

A graph G is k -color-critical (or k -critical) if $\chi(H) < \chi(G) = k$ for every proper subgraph H of G .

Characterization of 1-, 2-, 3-critical graphs.

Lower bounds

Simple lower bounds

$$\chi(G) \geq \omega(G)$$
$$\chi(G) \geq \frac{n(G)}{\alpha(G)}$$

Examples for $\chi(G) \neq \omega(G)$:

- **odd cycles** of length at least 5,

$$\chi(C_{2k+1}) = 3 > 2 = \omega(C_{2k+1})$$

- **complements of odd cycles** of order at least 5,

$$\chi(\overline{C}_{2k+1}) = k + 1 > k = \omega(\overline{C}_{2k+1})$$

- **random graph** $G = G(n, \frac{1}{2})$, almost surely

$$\chi(G) \approx \frac{n}{2 \log n} > 2 \log n \approx \omega(G)$$

Mycielski's Construction

The bound $\chi(G) \geq \omega(G)$ could be arbitrarily bad.

Construction. Given graph G with vertices v_1, \dots, v_n , we define supergraph $M(G)$.

$$V(M(G)) = V(G) \cup \{u_1, \dots, u_n, w\}.$$

$$E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}\}.$$

Theorem.

- (i) If G is triangle-free, then so is $M(G)$.
- (ii) If $\chi(G) = k$, then $\chi(M(G)) = k + 1$.

Upper bounds $\chi(G) \leq \Delta(G) + 1$.

Proof. Algorithmic. Greedy coloring.

Proposition $\chi(G) \leq \max_{H \subseteq G} \delta(H) + 1$.

Proof. Greedy coloring procedure with the ordering v_1, \dots, v_n , where v_i is a min-degree vertex of the graph $G[\{v_1, \dots, v_n\}]$.

Jordan Curves

A **curve** is a subset of \mathbb{R}^2 of the form

$$\alpha = \{\gamma(x) : x \in [0, 1]\} ,$$

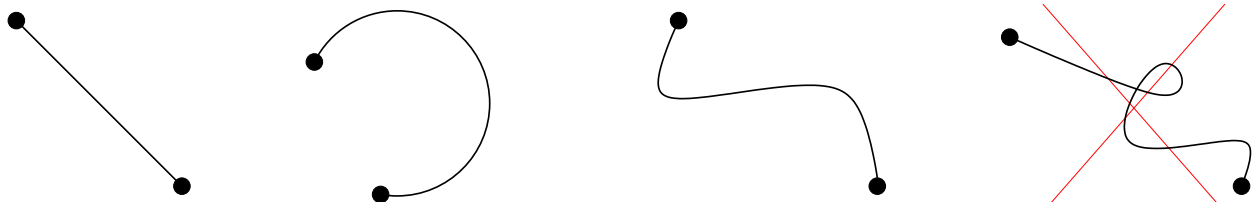
where $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ is a continuous mapping from the closed interval $[0, 1]$ to the plane. $\gamma(0)$ and $\gamma(1)$ are called the *endpoints* of curve α .

A curve is **closed** if its first and last points are the same. A curve is **simple** if it has no repeated points except possibly first = last. A closed simple curve is called a **Jordan-curve**.

Examples: Line segments between $p, q \in \mathbb{R}^2$

$$x \mapsto xp + (1 - x)q ,$$

circular arcs, Bezier-curves without self-intersection, etc...



Drawing of graphs

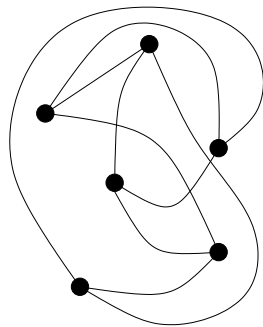
A **drawing** of a multigraph G is a function f defined on $V(G) \cup E(G)$ that assigns

- a point $f(v) \in \mathbb{R}^2$ to each vertex v and
- an $f(u), f(v)$ -curve to each edge uv ,

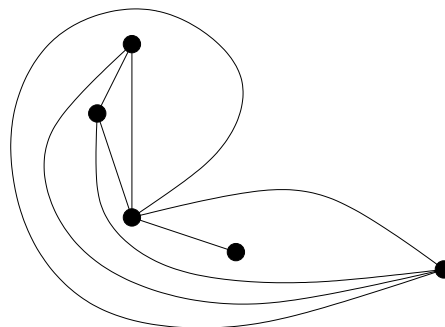
such that the images of vertices are distinct. A point in $f(e) \cap f(e')$ that is not a common endpoint is a **crossing**.

A multigraph is **planar** if it has a drawing without crossings. Such a drawing is a **planar embedding** of G . A planar (multi)graph *together* with a particular planar embedding is called a **plane (multi)graph**.

drawing



plane embedding



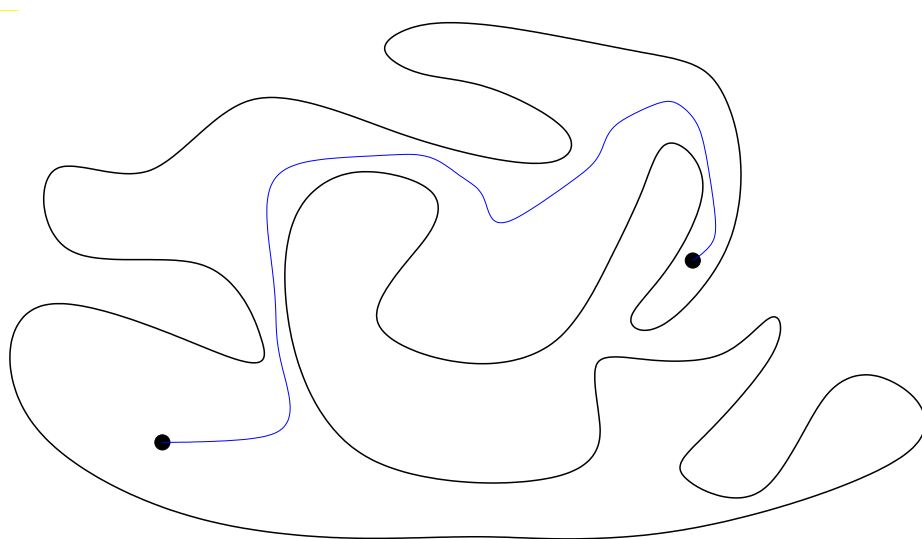
Are there non-planar graphs? _____

Proposition. K_5 and $K_{3,3}$ cannot be drawn without crossing.

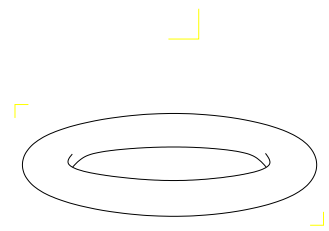
Proof. Define the *conflict graph* of edges.

The unconscious ingredient.

Jordan Curve Theorem. A simple closed curve C partitions the plane into exactly two faces, each having C as boundary.



Not true on the torus!

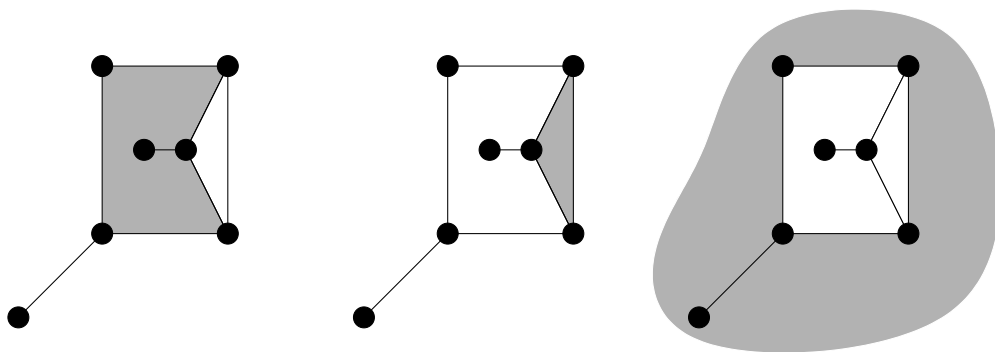




Regions and faces

An **open set** in the plane is a set $U \subseteq \mathbb{R}^2$ such that for every $p \in U$, all points within some small distance belong to U . A **region** is an open set U that contains a u, v -curve for every pair $u, v \in U$. The **faces** of a plane multigraph are the maximal regions of the plane that contain no points used in the embedding.

A finite plane multigraph G has one **unbounded face** (also called **outer face**).



Dual graph

Denote the set of faces of a plane multigraph G by $F(G)$ and let $E(G) = \{e_1, \dots, e_m\}$. Define the **dual** multigraph G^* of G by

- $V(G^*) := F(G)$
- $E(G^*) := \{e_1^*, \dots, e_m^*\}$, where the endpoints of e_i^* are the two (not necessarily distinct) faces $f', f'' \in F(G)$ on the two sides of e_i .

Remarks. Multiple edges and/or loops *could* appear in the dual of simple graphs

Different planar embeddings of the *same* planar graph could produce *different* duals.

Proposition. Let $l(F_i)$ denote the length of face F_i in a plane multigraph G . Then

$$2e(G) = \sum l(F_i).$$

Euler's Formula

Theorem.(Euler, 1758) If a plane multigraph G with k components has n vertices, e edges, and f faces, then

$$n - e + f = 1 + k.$$

Proof. Induction on e .

Base Case. If $e = 0$, then $n = k$ and $f = 1$.

Suppose now $e > 0$.

Case 1. G has a cycle.

Delete one edge from a cycle. In the new graph:

$$e' = e - 1, n' = n, f' = f - 1 \text{ (Jordan!)}, \text{ and } k' = k.$$

Case 2. G is a forest.

Delete a pendant edge. In the new graph:

$$e' = e - 1, n' = n, f' = f, \text{ and } k' = k + 1.$$

Remark. The dual may depend on the embedding of the graph, but the number of faces does *not*.

Coloring maps with 5 colors_____

Five Color Theorem. (Heawood, 1890) If G is planar, then $\chi(G) \leq 5$.

Proof. Take a minimal counterexample.

(i) There is a vertex v of degree at most 5.

(ii) Modify a proper 5-coloring of $G - v$ to obtain a proper 5-coloring of G . A contradiction.

Idea of modification: Kempe chains.

Coloring maps with 4 colors

Four Color Theorem. (Appel-Haken, 1976) For any planar graph G , $\chi(G) \leq 4$.

Idea of the proof.

W.l.o.g. we can assume G is a planar triangulation.

A **configuration** in a planar triangulation is a separating cycle C (the **ring**) together with the portion of the graph inside C .

For the Four Color Problem, a set of configurations is an **unavoidable set** if a minimum counterexample must contain a member of it.

A configuration is **reducible** if a planar graph containing it cannot be a minimal counterexample.

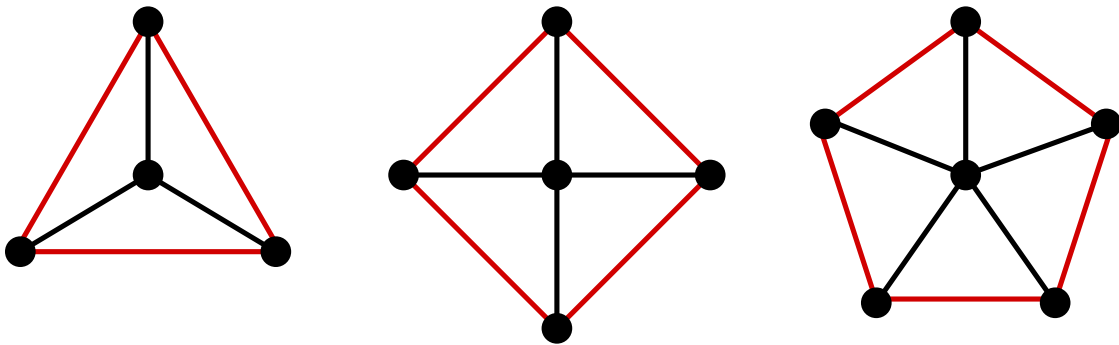
The usual proof attempts to

(i) find a set \mathcal{C} of unavoidable configurations, and

(ii) show that each configuration in \mathcal{C} is reducible.

Proof attempts of the Four Color Theorem

Kempe's original proof tried to show that the unavoidable set



is reducible.

Appel and Haken found an unavoidable set of 1936 of configurations, (all with ring size at most 14) and proved each of them is reducible. (1000 hours of computer time)

Robertson, Sanders, Seymour and Thomas (1996) used an unavoidable set of 633 configuration. They used 32 rules to prove that each of them is reducible. (3 hours computer time)

When is a graph planar? _____

Theorem(Euler, 1758) If a plane multigraph G with k components has n vertices, e edges, and f faces, then

$$n - e + f = 1 + k.$$

Corollary If G is a simple, planar graph with $n(G) \geq 3$, then $e(G) \leq 3n(G) - 6$.

If also G is triangle-free, then $e(G) \leq 2n(G) - 4$.

Corollary K_5 and $K_{3,3}$ are non-planar.

The **subdivision of edge** $e = xy$ is the replacement of e with a new vertex z and two new edges xz and zy . The graph H' is a **subdivision of H** , if one can obtain H' from H by a series of edge subdivisions. Vertices of H' with degree at least three are called **branch vertices**.

Theorem(Kuratowski, 1930) A graph G is planar **iff** G does not contain a subdivision of K_5 or $K_{3,3}$.