Bipartite graphs

A set of pairwise adjacent vertices in a graph is called a clique. A set of pairwise non-adjacent vertices in a graph is called an independent set.

A graph G is bipartite if $V(G)$ is the union of two independent sets of G. If these are disjoint, they are called the partite sets of G.

Examples. $K_{r,s}$ is bipartite, K_n is not bipartite for $n \geq 3$, P_n is bipartite for all $n \geq 1$, C_n is bipartite iff n is even (count edges leaving an independent set)

Example. The k-dimensional hypercube Q_k

$V(Q_k) = \{0, 1\}^k$

$E(Q_k) = \{xy : x$ and y differ in exactly one coordinate$\}$

Properties.

- $v(Q_k) = 2^k$
- Q_k is k-regular
- $e(Q_k) = k2^{k-1}$
- Q_k is bipartite

The beauty of being bipartite

Proposition. Let G be k-regular bipartite graph with partite sets A and B, $k > 0$. Then $|A| = |B|$.

Proof. Double count the edges of G by summing up degrees of vertices on each side of the bipartition.

Theorem. Every loopless multigraph G has a bipartite subgraph with at least $\frac{e(G)}{2}$ edges.

Proof by "extremality". (Consider a bipartite subgraph H with the *maximum number of edges* and prove that $d_H(v) \geq d_G(v)/2$ for every vertex $v \in V(G)$ (otherwise change H so to contradict its extremality. Finish with the Handshaking Lemma.))

Remark The constant multiplier $\frac{1}{2}$ of $e(G)$ in the Theorem is best possible.

Example: K_n. (for every bipartite $H \subseteq K_n$,

$$e(H) = i(n-i) \leq \left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor \right) = \left\lfloor \frac{n^2}{4} \right\rfloor$$

edges, which is $< (\frac{1}{2} + \epsilon)\binom{n}{2}$ for $\forall \epsilon > 0$ and large n.)

Walks, trails, paths, and cycles

A walk is an alternating list $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i.

A trail is a walk with no repeated edge.

A path is a walk with no repeated vertex.

A u, v-walk, u, v-trail, u, v-path is a walk, trail, path, respectively, with first vertex u and last vertex v.

If $u = v$ then the u, v-walk and u, v-trail is closed. A closed trail (without specifying the first vertex) is a circuit. A circuit with no repeated vertex is called a cycle.

The length of a walk trail, path or cycle is its number of edges.

Connectivity

G is connected, if there is a u, v-path for every pair $u, v \in V(G)$ of vertices.
Otherwise G is disconnected.

Vertex u is connected to vertex v in G if there is a u, v-path. The connection relation on $V(G)$ consists of the ordered pairs (u, v) such that u is connected to v.

Claim. The connection relation is an equivalence relation.

Lemma. Every u, v-walk contains a u, v-path.

The connected components of G are its maximal connected subgraphs (i.e. the equivalence classes of the connection relation).

An isolated vertex is a vertex of degree 0. It is a connected component on its own.

Cutting a graph

A cut-edge or cut-vertex of G is an edge or a vertex whose deletion increases the number of components.

If $M \subseteq E(G)$, then $G - M$ denotes the graph obtained from G by the deletion of the elements of M; $V(G - M) = V(G)$ and $E(G - M) = E(G) \setminus M$. Similarly, for $S \subseteq V(G)$, $G - S$ obtained from G by the deletion of S and all edges incident with a vertex from S.
For $e \in E(G)$, $G - \{e\}$ is abbreviated by $G - e$.
For $v \in V(G)$, $G - \{v\}$ is abbreviated by $G - v$.

Proposition. An edge e is a cut-edge iff it does not belong to a cycle.

Characterization of bipartite graphs

A bipartition of G is a specification of two disjoint independent sets in G whose union is $V(G)$.

Theorem. (König, 1936) A multigraph G is bipartite iff G does not contain an odd cycle.

Proof.

$\Rightarrow$ Easy.

$\Leftarrow$ Fix a vertex $v \in V(G)$. Define sets

$$A := \{w \in V(G) : \exists \text{ an odd } v, w\text{-path }\}$$

$$B := \{w \in V(G) : \exists \text{ an even } v, w\text{-path }\}$$

Prove that A and B form a bipartition.

Lemma. Every closed odd walk contains an odd cycle.

Proof. Strong induction.

Eulerian circuits

A multigraph is Eulerian if it has a closed trail containing all its edges. A multigraph is called even if all of its vertices have even degree.

Theorem. Let G be a connected multigraph. Then

G is Eulerian iff **G is even.**

Proof.

$\Rightarrow$ Easy.

$\Leftarrow$ Extremality: Consider longest trail T in G and prove that: (i) T is closed, (ii) $V(T) = V(G)$, (iii) $E(T) = E(G)$.

Eulerian trails

Theorem. A connected graph with exactly $2k$ vertices of odd degree decomposes into $\max\{k, 1\}$ trails.

Def. A decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.

Proof. Reduce it to the characterization of Eulerian graphs by introducing auxiliary edges.

Example. The "little house" can be drawn with one continous motion.

Remark. The theorem is "best possible", i.e. a decomposition into *less* than $\max\{k, 1\}$ trails is not possible.