Problem 5 Determine the characteristic polynomial of the sequence p_n, where p_n is the number of perfect matchings in the $3 \times (2n)$ grid graph G_n, defined as follows: $V(G_n) := [3] \times [2n]$ and

$$\begin{aligned} E(G_n) =& \{\{(i,j),(i,j+1)\} : i \in [3], j \in [2n-1]\} \\ &\cup \ \{\{(i,j),(i+1,j) : i \in [2], j \in [2n]\}. \end{aligned}$$

Give a closed formula for p_n.

We classify the perfect matchings according to the edge which contains the vertex $(2,2n)$. This vertex is covered either with a horizontal or vertical edge. In the first case, the squares $(1,2n)$ and $(3,2n)$ must also be covered by horizontal edges and we are left with a $3 \times (2n-2)$ grid graph G_{n-1}. Hence we have p_{n-1} matchings in this case. In the latter case, the edge covering $(2,2n)$ can either go upwards or downwards and in both cases the third vertex of the last column is forced to be covered by a horizontal edge. Both of these cases leave us with a graph H_n which is just the $3 \times (2n-1)$ grid graph with one corner vertex removed. Let us denote the number of perfect matchings of these by b_n. Hence

$$p_n = p_{n-1} + 2b_n \tag{1}$$

for every $n \geq 1$, where we set $a_0 = 1$.

To have a recursion for b_n, we classify the perfect matchings of H_n again according to the edge covering the middle vertex of the last column: it can either be vertical or horizontal. In the first case we are left with the graph G_{n-1} which has p_{n-1} perfect matchings. In the latter case we are forced to cover the other vertex of the last column by a horizontal edge as well, which in turn forces to cover the third vertex of the next-to-last column to be also covered by a horizontal edge. This leaves us with the graph H_{n-1}, which has b_{n-1} perfect matchings by definition. So altogether $b_n = p_{n-1} + b_{n-1}$ for every $n \geq 2$.

Substituting this to (1) we get $p_n = 3p_{n-1} + 2b_{n-1}$. We express $2b_{n-1}$ by using (1) for $n-1$ and get the recursion

$$p_n = 3p_{n-1} + (p_{n-1} - p_{n-2}) = 4p_{n-1} + p_{n-2}.$$

The initial conditions are $p_0 = 1$ and $p_1 = 3$.

So the characteristic polynomial is $p(x) = x^2 - 4x + 1$, which has 2 distinct roots $2 \pm \sqrt{3}$. From the theorem about homogenous linear recurrences with distinct roots in the characteristic polynomial from the lecture we know that there exists a closed formula of the form

$$p_n = c_1(2+\sqrt{3})^n + c_2(2-\sqrt{3})^n.$$

By solving this equation for $n = 0, 1$ we get $c_1 = \frac{1}{6}(3+\sqrt{3})$ and $c_2 = \frac{1}{6}(3-\sqrt{3})$ and thus

$$p_n = \frac{1}{6}(3+\sqrt{3})(2+\sqrt{3})^n + \frac{1}{6}(3-\sqrt{3})(2-\sqrt{3})^n.$$