Exercise 3

For the number of complete rooted binary trees b_n you can find the following recursion by looking at all possible sizes for the left an right subtree of the root

$$b_{n+1} = \sum_{i=0}^{n} b_i b_{n-i}$$

with initial conditions $b_0 = 0$ and $b_1 = 1$. This is the same recursion as for the Catalan numbers, but for the initial conditions. Define the generating funtion $b(x) = \sum_{n\geq 0} b_n x^n$.

$$b(x)^2 = (\sum_{n\geq 0} b_n x^n)^2 = \sum_{n\geq 0}(\sum_{i=0}^{n} b_i b_{n-i})x^n = \sum_{n\geq 1} b_{n+1}x^n = \frac{b(x)}{x} - 1$$

This gives us

$$b(x)^2 - \frac{b(x)}{x} + 1 = 0$$

and therefore

$$b(x) = \frac{1 \pm \sqrt{1-4x^2}}{2x}.$$

Since $b_0 = \lim_{x\to 0} b(x)$ the only solution is $b(x) = \frac{1-\sqrt{1-4x^2}}{2x}$. Now we obtain by the generalized binomial theorem

$$\begin{aligned}\frac{1-\sqrt{1-4x^2}}{2x} &= \frac{1-(1-4x^2)^{\frac{1}{2}}}{2x} = \frac{1-\sum_{n\geq 0}\binom{\frac{1}{2}}{n}(-4x^2)^n}{2x} \\ &= -\frac{1}{n}\sum_{n\geq 1}\binom{-\frac{1}{2}}{n-1}(-4)^{n-1}x^{2n-1} \\ &= \sum_{n\geq 1}\frac{1}{n}\binom{2n-2}{n-1}x^{2n-1} = \sum_{n\geq 1} C_{n-1}x^{2n-1},\end{aligned}$$

where C_n is the n-th Catalan number and the prelast inequality ist obtained from

$$r^{\underline{k}}(r-\frac{1}{2})^{\underline{k}}2^{2k} = (2r)^{\underline{2k}}$$

for $r = -\frac{1}{2}$ and $k = n-1$. This formula can be proofed by looking at $r^{\underline{k}}2^k$ which is the product of all even numbers between $2r-2k$ and $2r$ and $(r-\frac{1}{2})^{\underline{k}}2^k$ which is the product of all odd numbers in this interval. Together we get the product over all this numbers $(2r)^{\underline{2k}}$.

Finally we can read of the coefficient and get

$$b_n = \begin{cases} C_{\frac{n-1}{2}}, & \text{if } n \text{ is odd} \\ 0, & \text{otherwise.} \end{cases}$$