Graphs – Definition

A graph G is a pair $(V(G), E(G))$ consisting of

- a vertex set $V(G)$, and
- an edge set $E(G) \subseteq \binom{V(G)}{2}$.

If there is no confusion about the underlying graph we often just write $V = V(G)$ and $E = E(G)$.

Every graph is finite :) *

Model for: all sorts of networks (computer, road, transportation, social), relationships, job/applicant suitability; any situation with a binary relation

order of $G = v(G) := |V(G)|$
size of $G = e(G) := |E(G)|$

x and y are the endpoints of edge $e = \{x, y\}$.
They are called adjacent or neighbors.
e is called incident with x and y.

*in this course

Multigraphs, directed graphs

A loop is an edge whose endpoints are equal. Multiple edges have the same set of endpoints. In the definition of a "graph" we don't allow loops and multiple edges. To emphasize this, we often say "simple graph". When we do want to allow multiple edges or loops, we say multigraph.

Remarks A multigraph might have no multiple edges or loops. Every (simple) graph is a multigraph, but not every multigraph is a (simple) graph.

In a directed graph the edges are ordered pairs, that is $E \subseteq V^2$. On each edge $e = (x, y)$ one imagines a little arrow, pointing from the tail x of e to the head y of e.

In our course we do not deal much with directed graphs or multigraphs

Representations and special graphs

How to represent a graph?

to a human: (mostly) drawing

to a computer: adjacency matrix, incidence matrix

K_n is the complete graph on n vertices.

$K_{n,m}$ is the complete bipartite graph with partite sets of sizes n and m.

P_n is the path on n vertices

C_n is the cycle on n vertices

the length of a path or a cycle is its number of edges

Isomorphism of graphs

Actual “names” of vertices should not matter:

An isomorphism of G to H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ iff* $f(u)f(v) \in E(H)$. If there is an isomorphism from G to H, then we say G is isomorphic to H, denoted by $G \cong H$.

Claim. The isomorphism relation is an equivalence relation on the set of all graphs.

An isomorphism class of graphs is an equivalence class of graphs under the isomorphism relation.

Small examples. ...

Remark $K_n, K_{r,s}, P_n, C_n,$ etc ... are used ambiguously both for a concrete graph and the isomorphism class

Isomorphism of two large graphs is not easy to decide

*if and only if

Equivalence relation

A relation on a set S is a subset of $S \times S$.

A relation R on a set S is an equivalence relation if

1. $(x, x) \in R$ (R is reflexive)

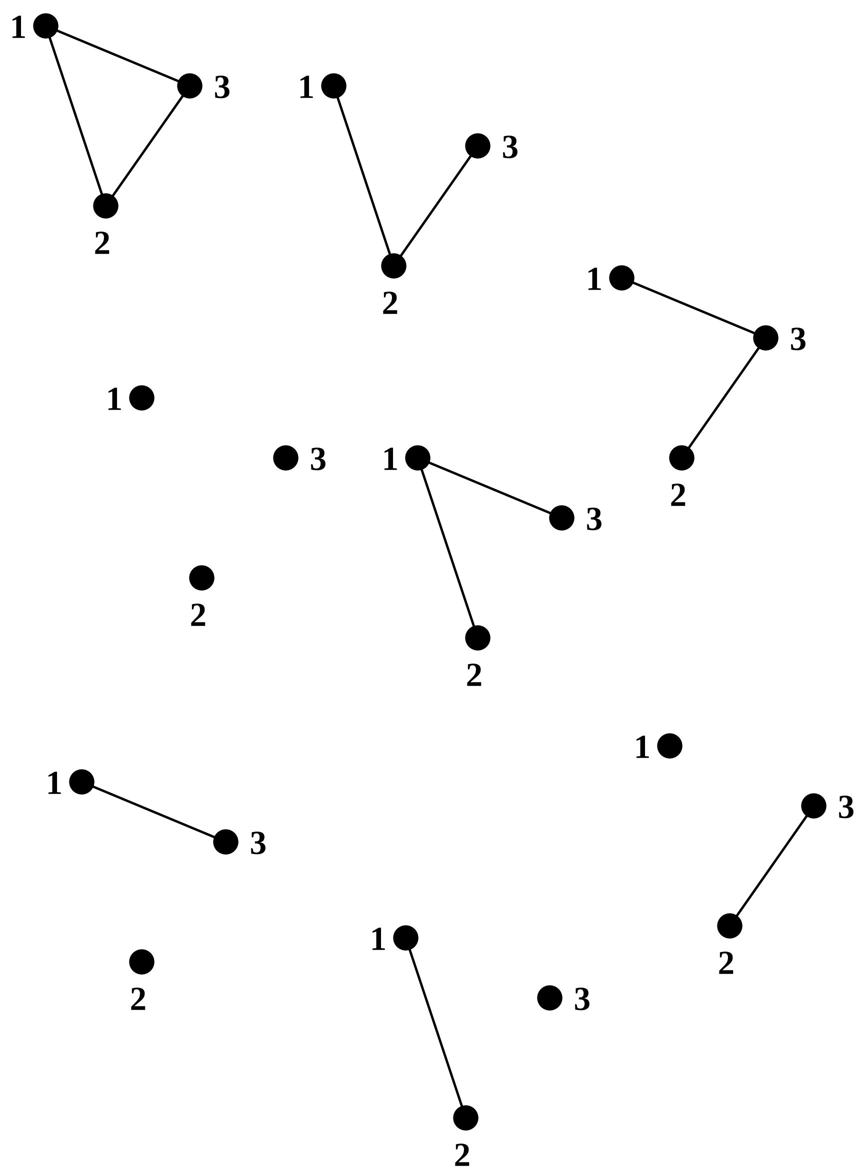
2. $(x, y) \in R$ implies $(y, x) \in R$ (R is symmetric)

3. $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$ (R is transitive)

An equivalence relation defines a partition of the base set S into equivalence classes. Elements are in relation iff they are within the same class.

Example. What are those graphs for which the adjacency relation is an equivalence relation?

Isomorphism classes



Automorphisms and the number of graphs

Remarks. labeled vs. unlabeled

“unlabeled graph” $\approx$ “isomorphism class”.

What is the number of labeled and unlabeled graphs on n vertices? $2^{\binom{n}{2}}$, $2^{\binom{n}{2}+O(n \ln n)}$

How large is the equivalence class of G? $\frac{n!}{|Aut(G)|}$

An automorphism of G is an isomorphism of G to G. A graph G is vertex transitive if for every pair of vertices u, v there is an automorphism that maps u to v.

Examples.

- Automorphisms of $K_n, P_n, C_n, K_{r,s}$
- Automorphisms of Petersen graph.

Neighborhoods and degrees

$N_G(v) = \{w \in V(G) : vw \in E(G)\}$ is the neighborhood of v in G

$d_G(v) = |N_G(v)|$ is the degree of a vertex v in G

Remark. The index G is suppressed when clear from context

$\Delta(G) = \max_{v \in V(G)} d(v)$ is the maximum degree of G

$\delta(G) = \min_{v \in V(G)} d(v)$ is the minimum degree of G

G is regular if $\Delta(G) = \delta(G)$

G is k-regular if the degree of each vertex is k.

Examples: $K_n, K_{r,s}, P_n, C_n$

“The” counterexample: the Petersen graph

Petersen graph P**:** $V(P) = \binom{[5]}{2}$

$$E(P) = \{\{A, B\} : A \cap B = \emptyset\}$$

Properties.

- P is 3-regular
- adjacent vertices have no common neighbor
- non-adjacent vertices have exactly one common neighbor

Corollary. The girth of the Petersen graph is 5.

Def. The girth of a graph is the length of its shortest cycle

Degrees and the Handshaking Lemma

Is there a graph with degree sequence $1, 2, 2, 3, 3, 4, 4, 5$? And degree sequence $1, 2, 2, 3, 3, 4, 4, 5, 5, 6$?

Handshaking Lemma. For any graph G,

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

Corollary. Every graph has an even number of vertices of odd degree.
No graph of odd order is regular with odd degree.

Corollary. In a graph G the average degree is $\frac{2e(G)}{v(G)}$ and hence $\delta(G) \leq \frac{2e(G)}{v(G)} \leq \Delta(G)$.

Corollary. A k-regular graph with n vertices has $kn/2$ edges.

Example. $e(P) = \frac{3 \cdot 10}{2} = 15$

Complements and (induced) subgraphs

The complement $\overline{G}$ of a graph G is a graph with

- vertex set $V(\overline{G}) = V(G)$ and
- edge set $E(\overline{G}) = \binom{V}{2} \setminus E(G)$.

A graph is self-complementary if it is isomorphic to its complement.

Example. P_4, C_5

H is a subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. We write $H \subseteq G$. We also say G contains H and write $G \supseteq H$.

Example. $P_n \subseteq C_n \subseteq K_n \subseteq K_{n+1}$

For a subset $S \subseteq V(G)$ define $G[S]$, the induced subgraph of G on S: $V(G[S]) = S$ and
$E(G[S]) = \binom{S}{2} \cap E(G)$.

Example. C_n does not have an induced subgraph isomorphic to P_n, but C_{n+1} does.