Exercise 1

Let $0 < k \leq l \leq m$ be integers. We want to construct a graph G with vertex-connectivity $\kappa(G) = k$, edge-connectivity $\kappa'(G) = l$ and minimum degree $\delta(G) = m$.

Start with two disjoint copies of K_{m+1} on vertex sets V_1,V_2. Choose two sets $A = \{a_1, \dots, a_k\} \subseteq V_1$ and $B = \{b_1, \dots, b_k\} \subseteq V_2$ of k vertices and connect them using l edges such that all edges of the form $a_i b_i$ are there (and the remaining $l - k$ edges are arbitrary between A and B).

The degree of every vertex is at least m, since every vertex is contained in one of the K_{m+1}. Furthermore, since $k < m + 1$, there is a vertex which is not contained in any of the l crossing edges, so the minimum degree is exactly m.

Either of the two sets A,B is a vertex cut of size k, so the connectivity is at most k. Assume there is a vertex cut S of size $k - 1$. After removing S both K_{m+1}'s remain connected. Moreover since $|S| = k - 1$ there exists an $i \in [k]$ such that $a_i \notin S$ and $b_i \notin S$, so the edge $a_i b_i$ connects the remainder of the two cliques. Thus there is no vertex cut of size $k - 1$ and hence the connectivity is exactly k.

The edge connectivity is at most l, since $[V_1, \overline{V_1}]$ is an edge cut of size l. For any other edge cut $[S, \bar{S}]$ there exists an $i \in \{1, 2\}$ such that $\emptyset \neq S \cap V_i \neq V_i$ and therefore $|[S, \bar{S}]| \geq |S \cap V_i|(m + 1 - |S \cap V_i|) \geq m \geq l$, so the edge-connectivity is exactly l.

Exercise 2

(a) Let G be a k-connected graph. Let G' be a graph obtained from G by adding a new vertex v with at least k neighbours.

Assume for a contradiction that there is a vertex cut S of size at most $k - 1$ in G'.

Case 1: If $v \notin S$ then $G'[V \setminus S] = G - S$ is still connected since there is no vertex cut of size at most $k - 1$ in G. The vertex v still has at least one neighbour left in $G - S$, so the whole $G' - S$ is connected.

Case 2: If $v \in S$ then $S \setminus \{v\}$ is a vertex cut of size $k - 2$ of G, which is a contradiction.

So there is no vertex cut of size at most $k - 1$ in G', that is, G' is k-connected.

(b) Let G be a 2-connected graph and $e_1 = a_1 a_2$, $e_2 = b_1 b_2$ be two distinct edges in G.

We construct a graph G' by adding two new vertices x and y to G, and edges from x to a_1 and a_2 and edges from y to b_1 and b_2.

By the first part of this exercise we know that G' is also 2-connected, so by Whitney's theorem there exist two internally disjoint x, y paths in G'. The union of these two paths is a cycle C that uses the incident edges $a_1 x$ and $x a_2$ as well as the incident edges $b_1 y$ and $y b_2$. Replacing these pairs of edges by the edges $a_1 a_2$ and $b_1 b_2$ "short-cuts" C and creates a cycle in G which contains both of the edges e_1 and e_2.

Exercise 3

" $\Leftarrow$ " Assume first that for every ordered triple of distinct vertices (x, y, z) there exists an x, z-path through y. In particular G is connected. For a contradiction, suppose that there exists a cut-vertex z and consider any two vertices x, y in different connected components of $G - z$.

By our assumption there must exist an x, z-path P through y in G. However, the part of P between x and y is an x, y-path not containing z. This is a contradiction to the assumption that x and y lie in different components of $G - z$. Hence G has no cut-vertex and is 2-connected.

" $\Rightarrow$ " Let G now be 2-connected and let (x, y, z) be an ordered triple of distinct vertices. By Whitney's Theorem there exists two internally disjoint y, z-paths R_1 and R_2 and also an x, y-path Q. Let w be the first vertex on Q (starting from x) which is also contained in $V(R_1) \cup V(R_2)$. Note that there exists such a vertex as $y \in V(Q) \cap (V(R_1) \cup V(R_2))$.

Assume without loss of generality that $w \in V(R_1)$. Now we construct the required path P. First let us take the part of Q from x to w, then take the part of R_1 from w to y and finally the whole R_2 from y to z. The intersection of the first part with last two is only w, because w was the *first* vertex on Q which is also in $V(R_1) \cup V(R_2)$. The intersection of the second and third parts is only y, because R_1 and R_2 were internally disjoint.

Hence P is an x, z-path through y, as required.

Exercise 4

We prove the statement by induction on the number n of vertices in G. If $n = 1$ then $G = K_1$ and $\chi(G) + \chi(\overline{G}) = 1 + 1$, so the base case is fine.

Let $n > 1$. Take an arbitrary vertex $v \in V(G)$, delete it from G and apply induction for $G' = G - v$. By definition $\overline{G - v} = \overline{G} - v$, so

$$\chi(G - v) + \chi(\overline{G} - v) \leq n - 1 + 1 = n.$$

Clearly, $\chi(G) \leq \chi(G - v) + 1$ and $\chi(\overline{G}) \leq \chi(\overline{G} - v) + 1$, since one could always create a proper coloring of G (or $\overline{G}$) by taking an optimal coloring of $G - v$ (or $\overline{G} - v$) and add a new color to v. So

$$\chi(G) + \chi(\overline{G}) \leq \chi(G - v) + 1 + \chi(\overline{G} - v) + 1 \leq n + 2.$$

Since all the numbers involved are integers, if *any* of the inequalities above are strict, we proved the required upper bound of $n+1$. Hence we are done otherwise $\chi(G) = \chi(G-v)+1$ and $\chi(\overline{G}) = \chi(\overline{G}-v)+1$, as well as $\chi(G-v)+\chi(\overline{G}-v) = n$.

Note, however that if indeed $\chi(H) = \chi(H - u) + 1$ for some graph H and vertex $u \in V(H)$, then $d(u) \geq \chi(H - u)$, since if $|N(u)| \leq \chi(H - u) - 1$ then an optimal coloring of $H - u$ would be possible to extend to u by simply taking any color which does not appear on the neighbors of u.

This observation provides contradiction with the three equalities above. From the first two equations we get $d_G(v) \geq \chi(G-v)$ and $d_{\overline{G}}(v) \geq \chi(\overline{G}-v)$, implying

$$n-1 = d_G(v) + d_{\overline{G}}(v) \geq \chi(G-v) + \chi(\overline{G}-v)$$

and contradicting the third equation.

Exercise 5

A graph G is k-colour-critical if $\chi(H) < \chi(G) = k$ for every proper subgraph H of G. Let $M(G)$ be the Mycielski of a k-colour-critical graph G on vertex set $V(G) = \{v_1, \dots, v_n\}$. That is a graph with $V(M(G)) = V(G) \cup \{u_1, \dots, u_n, w\}$ and $E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}\}$. We know from the lecture that $\chi(G) = k$ implies $\chi(M(G)) = k+1$.

Since there are no isolated vertices in $M(G)$, it is enough to check that $M(G) - e$ is k-colorable for every edge $e \in E(M(G))$. There are three cases.

Case 1: $e = v_i v_j$ for some $1 \leq i < j \leq n$. Since G is color-critical, we can color $G - e$ properly with $k-1$ colors, say 1 up to $k-1$. Then, we color the vertices $u_1, \dots, u_n$ with color k, and color w with color 1. This is a proper k-coloring of $M(G) - e$.

Case 2: $e = v_i u_j$ for some $1 \leq i \neq j \leq n$. By the definition of Mycielski's construction, we have $v_i v_j \in E(G)$. Now, consider $H = G - v_i v_j$. Since G is k-color-critical, H is $(k-1)$-colorable. So, $M(H)$ is k-colorable by the theorem in the lecture. Moreover, we can see that $M(G) - e = M(H) + v_i v_j + v_j u_i$.

Now, the idea is to color $M(H)$ first by k colors properly, and modify this coloring into a proper k-coloring of $M(G) - e$. Here is an explicit method. First we color V by $k-1$ colors, say 1 up to $k-1$, so that this will be a proper $(k-1)$-coloring of H. Then, for each $\ell \in \{1, \dots, n\}$, color $u_\ell \in U$ by the color used for $v_\ell \in V$. Finally, we color w by the color k. This is a proper k-coloring of $M(H)$.

Now, we add $v_i v_j$ and $v_j u_i$ to $M(H)$ so that the result will be $M(G) - e$. Then, we change the color of v_j to the color k. Since the color k was not used in $U \cup V$, this coloring is still proper. Thus, we obtained a proper k-coloring of $M(G) - e$.

Case 3: $e = u_i w$ for some $i = 1, \dots, n$. First, consider the graph $G - v_i$. Since G is color-critical, $G - v_i$ is $(k-1)$-colorable. Now, in $M(G) - e$, we color $V \setminus \{v_i\}$ by $k-1$ colors, say 1 up to $k-1$, according to a proper $k-1$-coloring of $G - v_i$. Next, for each $\ell \in \{1, \dots, n\} \setminus \{i\}$ we color u_ℓ by the color used for v_ℓ. Then, we can color v_i, u_i, w by the color k. We can see that this is a proper k-coloring of $M(G) - e$ because v_i, u_i, w form an independent set in $M(G) - e$, and the color k is not used on the other vertices.

To summarize, in each of the three cases, we have obtained a proper k-coloring of $M(G) - e$, showing that Mycielski's construction preserves color-criticality.

Exercise 6

Let G be a k-chromatic graph. Take a proper k-colouring of G. For every pair of colours i and j there exists an edge with adjacent vertices coloured with i and j. Indeed, otherwise we could combine the vertices of colour i and j into a single colour class, resulting in a proper $(k-1)$-colouring, which is in contradiction with $\chi(G) = k$. There are $\binom{k}{2}$ possible pairs of distinct colours, giving us at least $\binom{k}{2}$ edges in G.

Let G be contained in the union of m copies of K_m (not necessarily edge- or vertex-disjoint). This implies $e(G) \leq m\binom{m}{2}$. Let k be the chromatic number of G. Then by the above $\binom{k}{2} \leq e(G)$. Putting the two inequalities together we obtain $\binom{k}{2} \leq m\binom{m}{2}$, which is equivalent to $k^2 - k < m^3 - m^2$. This implies $k^2 < m^3$.