Connectivity

A separating set (or vertex cut) of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. For $G \neq K_n$, the connectivity of G is $\kappa(G) := min\{|S| : S \text{ is a vertex cut}\}$. By definition, $\kappa(K_n) := n - 1$. A graph G is k-connected if there is no vertex cut of size $k - 1$. (i.e. $\kappa(G) \geq k$)

Examples. $\kappa(K_{n,m}) = \min\{n, m\}$
$\kappa(Q_d) = d$

Edge-connectivity

An edge cut of a multigraph G is an edge-set of the form $[S, \bar{S}]$, with $\emptyset \neq S \neq V(G)$ and $\bar{S} = V(G) \setminus S$.

For $S, T \subseteq V(G)$, $[S, T] := \{xy \in E(G) : x \in S, y \in T\}$.

The edge-connectivity of G is

$$\kappa'(G) := \min\{\ |[S, \bar{S}]|\ : [S, \bar{S}] \text{ is an edge cut}\}.$$

A graph G is k-edge-connected if there is no edge cut of size $k-1$ (i.e. $\kappa'(G) \geq k$).

Theorem. (Whitney, 1932) If G is a simple graph, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Homework. Example of a graph G with $\kappa(G) = k$, $\kappa'(G) = l$, $\delta(G) = m$, for any $0 < k \leq l \leq m$.

Characterization of 2-connected graphs

Theorem. (Whitney,1932) Let G be a graph, $n(G) \geq 3$. Then G is 2-connected iff for every $u, v \in V(G)$ there exist two internally disjoint u, v-paths in G.

Menger's Theorem

Given $x, y \in V(G)$, a set $S \subseteq V(G) \setminus \{x, y\}$ is an x, y-separator (or an x, y-cut) if $G - S$ has no x, y-path.

A set $\mathcal{P}$ of paths is called pairwise internally disjoint (p.i.d.) if for any two path $P_1, P_2 \in \mathcal{P}$, P_1 and P_2 have no common internal vertices.

Define

$$\kappa(x,y) := \min\{|S| : S \text{ is an } x,y\text{-cut,}\} \text{ and}$$
$$\lambda(x,y) := \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.i.d. } x,y\text{-paths}\}$$

Local Vertex-Menger Theorem (Menger, 1927) Let $x, y \in V(G)$, such that $xy \notin E(G)$. Then

$$\kappa(x,y) = \lambda(x,y).$$

Corollary (Global Vertex-Menger Theorem) A graph G is k-connected iff for any two vertices $x, y \in V(G)$ there exist k p.i.d. x, y-paths.

Proof: **Lemma.** For every $e \in E(G)$, $\kappa(G - e) \geq \kappa(G) - 1$.