Matchings

A matching is a set of (non-loop) edges with no shared endpoints. The vertices incident to an edge of a matching M are saturated by M, the others are unsaturated. A perfect matching of G is matching which saturates all the vertices.

Examples. $K_{n,m}$, K_n, Petersen graph, Q_k; graphs without perfect matching

A maximal matching cannot be enlarged by adding another edge.

A maximum matching of G is one of maximum size.

Example. Maximum $\neq$ Maximal

Characterization of maximum matchings

Let M be a matching. A path that alternates between edges in M and edges not in M is called an M-alternating path.
An M-alternating path whose endpoints are unsaturated by M is called an M-augmenting path.

Theorem(Berge, 1957) A matching M is a maximum matching of graph G iff G has no M-augmenting path.

Proof. ($\Rightarrow$) Easy.
($\Leftarrow$) Suppose there is no M-augmenting path and let M^* be a matching of maximum size.
What is then $M \triangle M^*$???

> **Lemma** Let M_1 and M_2 be matchings of G. Then each connected component of $M_1 \triangle M_2$ is a path or an even cycle.
>
> For two sets A and B, the symmetric difference is $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Hall's Condition and consequences

Theorem (Marriage Theorem; Hall, 1935) Let G be a bipartite (multi)graph with partite sets X and Y. Then there is a matching in G saturating X iff $|N(S)| \geq |S|$ for every $S \subseteq X$.

Proof. ($\Rightarrow$) Easy.

($\Leftarrow$) Not *so* easy. Find an M-augmenting path for *any* matching M which does not saturate X.
(Let U be the M-unsaturated vertices in X. Define

$$\begin{aligned} T &:= \{y \in Y : \exists\ M\text{-alternating } U, y\text{-path}\}, \\ S &:= \{x \in X : \exists\ M\text{-alternating } U, x\text{-path}\}. \end{aligned}$$

Unless there is an M-augmenting path, $S \cup U$ **violates** Hall's condition.)

Corollary. (Frobenius (1917)) For $k > 0$, every k-regular bipartite (multi)graph has a perfect matching.

Application: 2-Factors

A factor of a graph is a spanning subgraph. A k-factor is a spanning k-regular subgraph.

Every regular bipartite graph has a 1-factor.

Not every regular graph has a 1-factor.

But...

Theorem. (Petersen, 1891) Every $2k$-regular graph has a 2-factor.

Proof. Use Eulerian cycle of G to create an auxiliary k-regular bipartite graph H, such that a perfect matching in H corresponds to a 2-factor in G.

Graph parameters

The size of the largest matching (*independent set of edges*) in G is denoted by $\alpha'(G)$.

A vertex cover of G is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. (The vertices in Q *cover* $E(G)$).
The size of the smallest vertex cover in G is denoted by $\beta(G)$.

Claim. $\beta(G) \geq \alpha'(G)$.

Certificates

Suppose we knew that in some graph G with 1121 edges on 200 vertices, a particular set of 87 edges is (one of) the largest matching one could find. How could we convince somebody about this?

Once the particluar 87 edges are shown, it is easy to check that they are a matching, indeed.

But why isn't there a matching of size 88? Verifying that none of the $\binom{1121}{88}$ edgesets of size 88 forms a matching could take some time...

If we happen to be so lucky, that we are able to exhibit a vertex cover of size 87, we are saved. It is then reasonable to check that all 1121 edges are covered by the particular set of 87 vertices.

Exhibiting a vertex cover of a certain size **proves** that no larger matching can be found.

Certificate for bipartite graphs

1. Correctness of the certificate:

A vertex cover $Q \subseteq V(G)$ is a certificate proving that no matching of G has size larger than $|Q|$.
That is: $\beta(G) \geq \alpha'(G)$, valid for every graph.

2. Existence of optimal certificate for bipartite graphs:

Theorem. (König (1931), Egerváry (1931))
If G is bipartite then $\beta(G) = \alpha'(G)$.

Remarks

1. König's Theorem $\Rightarrow$ For bipartite graphs there always exists a vertex cover proving that a particular matching of maximum size is really maximum.

2. This is NOT the case for general graphs: C_5.

Proof of König's Theorem: For any minimum vertex cover Q, apply Hall's Condition to match $Q \cap X$ into $Y \setminus Q$ and $Q \cap Y$ into $X \setminus Q$.