Material covered in class

Discrete Mathematics I — Summer 2014

Week 1: (Aigner, Chapter 1) Notation: $[n], \mathbb{N}, \binom{X}{k}, \binom{n}{k}, 2^X, X^{\underline{k}}, n^{\underline{k}}$. Elementary counting principles: Rule of Sum (basis of every proper case analysis (cases should be **complete** and **exclusive**); examples: colored pairs of socks in the drawer, Pascal recurrence (Pascal triangle)), Rule of Product (examples: number of bit-sequences of length n) Generalized Rule of Product (k-permutations of a set X (injective functions from $[k]$ to X), value of $\binom{n}{k}$, number of ways to seat n people for chess (the importance of **knowing precisely** what we want to count (here the unordered pairings of an n-element set (two proofs))), Rule of Bijection (subsets of an n-element set vs. bit-sequences of length n, k-permutations of an n element set vs. $\binom{[n]}{k} \times [k]^k$), Rule of Double Counting (formula for the sum of the first n positive integers, average behaviour of the function $d(n)$ representing the number of divisors of n), Combinatorial proofs of identities ($\binom{n}{k} = \binom{n}{n-k}$, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$), Binomial Theorem and its corollaries (number of odd and even subsets is equal (another proof via bijection), distributing 31 pennies between 3 children (proof by introducing dividers), Generalization: k-multisets over a set X, number of k-multisets of an n-element set (proof by bijection)

Week 2: (Aigner, Chapter 1) Binomial identities ($\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}$ and Vandermonde identity), Falling factorial polynomial, Rising factorial polynomial, Binomial coefficient for complex numbers, Polynomial Method, Reciprocity ($(-x)^{\underline{k}} = (-1)^k x^{\overline{k}}$), Pascal recurrence and Vandermonde identity for polynomials, set partitions, Bell numbers with recurrence, Stirling numbers of second kind (examples, explicit values for $S_{n,1}, S_{n,2}, S_{n,n-1}, S_{n,n}$), Recurrence for Stirling numbers of second kind (combinatorial proof), Stirling matrix, Polynomial identity ($x^n = \sum_{k=0}^{n} S_{n,k} x^{\underline{k}}$)

Week 3: (Aigner, Chapter 1) proof of Stirling recurrence using polynomials. What is the inverse of the Stirling matrix of the second kind? Def: symmetric group S_n, word representation and cycle decomposition of a permutation, fixed point of permutation, transposition, cyclic permutation, number of cyclic permutations, Stirling number of the first kind $s_{n,k}$, Examples: $s_{n,1}, s_{n,n}, s_{n,n-1}$, $\sum_k s_{n,k} = n!$, recurrence: $s_{n,k} = s_{n-1,k-1} + (n-1)s_{n-1,k}$, $x^{\underline{n}} = \sum_k (-1)^{n-k} s_{n,k} x^k$, Corollary: the matrix of the signed Stirling numbers of the first kind is the inverse of the matrix of the Stirling numbers of the second kind; for all $i, j \geq 0$, $\sum_{k=j}^{i} (-1)^{k-j} S_{i,k} s_{k,j} = \delta_{i,j}$ (Kronecker-delta); The twelvefold way of counting, number-partitions of n,

$p(n), p(n;k), p(n;k;m), p_d(n)$, etc... sets $Par(n), Par(n;k), Par(n;k;m), Par_d(n)$, etc ... Example: $n = 6$, small values of matrix, recurrence: $p(n;k) = p(n-1;k-1) + p(n-k;k)$, Prop: $p(n;k;m) = p(n;m;k)$, Ferrers diagram, conjugate partition, Prop: $p(\leq nm; \leq n; \leq m) = \binom{n+m}{m}$, bijection between Ferrers diagrams and lattice paths and then between lattice path and sequences of directions, Prop: $p_d(n;k;\leq m) = p(n-\binom{k+1}{2}; \leq k; \leq m-k)$, HW: $p_d(n)$ is the number of partitions into odd terms, Generating functions: number of ways to assemble n cents if we have n_1 one cent coins, n_2 two cents coins, and n_3 five cents coins, expressed as the coefficient of a polynomial,

Week 4: (Matousek-Nesteril, Chapter 12, Aigner, Chapter 3.1) encoding sequences as power series, series of constant 1 series, Proposition about absolute convergence around 0, generating function of a sequence, Examples: $-\ln(1-x), e^x, (1+x)^r$, Fibonacci sequence, explicit formula via its generating function and its partial fraction decomposition, operations with sequences and their generating functions (linear combination, shift to the right/left, substituting $\alpha x, x^n$, Example: $a_i = 2^{\lfloor i/2 \rfloor}$, differentiation/integration, Example: $a_i = (i+1)^2$, product, Example: using derivative and product to derive combinatorial formulas) General theorem about solution of homogeneous linear recurrences (proof of the case of distinct roots (Vandermonde determinant), general case HW), Application: digits of $(\sqrt{2}+\sqrt{3})^{1980}$ (simultaneous recurrences), generating function for the number of binary trees (formal definition of binary trees)

Week 5: (Matousek-Nesteril, Chapter 12, 3) Catalan numbers (recursion, generating function, precise formula), Exponential generating functions (example: number of involutions), Precise vs asymptotic counting, (Example: ugly formula), asymptotic notation $(O(f(n)), o(f(n)), \Omega(f(n)), \Theta(f(n)), \sim, \ll, \gg$, asymptotic hierarchy of functions (log powers, polynomials, exponentials, etc ...) Estimates: simple to more and more involved: sum of cubes, factorial (Stirling formula (without proof)), estimates on binomial coefficients ($\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$), number of ordered partitions of integers is 2^{n-1}, upper estimate on number of unordered partitions of integers (giving the correct order of magnitude (with larger constant factor).

Week 6: (Matousek-Nesteril, Chapter 12, van Lint Wilson,) Catalan numbers and lattice paths, (a combinatorial proof of the formula) Inclusion/Exclusion Formula (counting the complement of a set, counting positive integers relative prime to 30, general formula (proof by polynomial identity $\prod_{i=1}^{k}(1+x_i) = \sum_{I \subseteq [k]} \prod_{i \in I} x_i$ and the characteristic functions of the sets), Applications: derangements, formula for Euler's totient function. $\sum_{d|n} \phi(d) = n$, Möbius function on $\mathbb{N}$, Möbius inversion, proof of the formula for $\phi(n)$ via Möbius inversion, $\sum_{d|n} \mu(d)$, formula for the number of cyclic arrangements of $0s$ and 1s on unlabeled cyclic positions (proof via Möbius inversion)

Week 7: (Aigner: Chapter 5, Matousek-Nesetril, Brualdi) locally finite poset, interval, incidence algebra over field $\mathbb{F}$. convolution product, product is associative, Kronecker delta is the unique identity element, there is a unique inverse to every f for which $f(x,x) \neq 0$ for every $x \in P$ (HW), zeta function, Möbius function of the poset, Möbius Inversion (from below and from above), special cases (HW), Pigeonhole Principle (among three ordinary people ..., strains of hair in Berlin) any 101-subset of the first 200 positive integers contains two that divide each other, Chinese Remainder Theorem, Fruit basket with apples, bananas, and oranges, General form of Pigeonhole Principle, a special case: "averaging" (if a set of Q elements is partitioned into n disjoint sets then one of these partitioning sets contains AT LEAST $\lceil Q/n \rceil$ elements. (and one the partitioning sets contains AT MOST $\lfloor Q/n \rfloor$ elements)); Pigeonhole Principle only gives an EXISTENCE PROOF, no clever general method (algorithm) to find the special pigeonhole fast; Application: Erdős-Szekeres Theorem (THERE IS a sequence of length n such that the longest increasing (and the longest decreasing) subsequence is of length $\lceil \sqrt{n} \rceil$; EVERY sequence of n distinct real numbers contains a monotone subsequence of length $\lceil \sqrt{n} \rceil$.)

Week 8: (Jukna, Ch. 27) $2k-1$ elements are enough to find a monochromatic subset of size k in any two-coloring of the $2k-1$ elements. How many elements do we need if we two-color not the one-, but the two-element subsets? 5 elements is not enough to find a monochromatic 3-set. Proposition: 6-elements are enough. (For any two-coloring of the two-element subsets of a six-element set there is a three-element subset whose pairs all have the same color.) Definition: graph G, vertices, edges, order of G, size of G, x and y are adjacent, neighbors, complete graph K_n. Ramsey number $R(k)$. Ramsey's Theorem: $R(k)$ is finite for every $k \in \mathbb{N}$. (In proof $R(k) \leq 4^k$). Known bounds for $R(4), R(5), R(6), R(10)$ (without proof). Lower bound construction with $(k-1)^2$ vertices. Exponential lower bound (Theorem (Erdős): $\sqrt{2}^k < R(k)$, "start of the probabilistic method"), Definition: asymmetric Ramsey number $R(k,\ell)$, Improved upper bound on $R(k) = R(k,k)$ ($R(k,\ell) \leq R(k,\ell-1) + R(k-1,\ell)$, Corollary: $R(k,k) = O(4^k/\sqrt{k})$), Open Problems (\$500 each) about $\lim_{k\to\infty} \sqrt[k]{R(k,k)}$, Not known whether

$R(k,k) > 1.4143^k$ or $R(k,k) < 3.9999^k$. Definition: graph, vertex set, edge set, (Model for networks (computer, road, transportation, social), relationships in a community, job/applicant suitability; any situation where a binary relation plays a role; we mostly talk about simple graphs but there are also multigraphs (multiple edges, loops) and directed graphs (where edges have a "direction", they are not sets but ordered pairs), order of G, size of G, x and y are adjacent, neighbors, $x \in V$ and $e \in E$ are incident; Representing graphs (drawing, adjacency matrix), Special graphs: complete graph K_n, path P_n of length $n-1$, cycle C_n of length n (length is the number of edges of a path or a cycle), complete bipartite graph $K_{n,m}$; Definition: Isomorphism of graphs ("the name of the vertices is not important"), Example of isomorph and non-isomorph graphs (some invariants of graphs under isomorphism), number of labeled graphs is $2^{\binom{n}{2}}$, symmetries (automorphisms) of graphs, number of automorphisms, Example: $K_n, P_n, C_n, K_{n,m}$.

Week 9: (West: Chapter 1) asymptotic estimate for the number of unlabeled graphs (isomorphism classes) on n vertices, Def: neighborhood, degree of a vertex, k-regular graph, Example: Petersen graph P, Prop.: Petersen is 3-regular, every two adjacent vertices have no common neighbor, every two nonadjacent vertices have exactly one common neighbor, Corollary: girth of Petersen is five (Definition of girth); Is there a graph with degree sequence $2,3,3,4,4,5,5,6$? YES And $2,3,3,4,4,5,5,6,6,7$? NO; Handshake Lemma: For all graphs G, $\sum_{v \in V(G)} d(v) = 2e(G)$; Corollary 1: The number of vertices of odd degree is even in every graph; Corollary 2: Number of edges in a k-regular graph on n vertices is $\frac{kn}{2}$. Example: $e(P) = \frac{3 \cdot 10}{2} = 15$. Def: complement of G, Example: C_5 is self-complementary, H is subgraph of G ($H \subseteq G$), G contains H ($G \supseteq H$), Example: $P_n \subseteq C_n \subseteq K_n \subseteq K_{n+1}$, Def: H is induced subgraph of G, Example: P_n is not induced subgraph of C_n, but induced subgraph of C_{n+1}, Def: clique, independent set, bipartite graph (bipartition, partite set), Example: $K_{n,m}$ is bipartite, K_n is not bipartite for $n \geq 3$, P_n is bipartite, C_n is bipartite if and only if n is even (Proof of "only if" direction by counting edges leaving an independent set), Example: Hypercube Q_n, Proposition: Q_n is bipartite for every n. Proposition: If G is a regular bipartite graph of positive degree, then its partite sets have equal size.

Week 10: (West Chapter 1) Thm: Every graph has a bipartite subgraph containing half of its edges (proof by extremality), Definitions: u,v-walk, -trail, -path, closed walk, circuit, cycle in a graph, length, G is connected, connected component: maximal connected subgraph, "connectedness" relation $\sim_c$ on $V(G)$ is an equivalence relation (transitivity: every u, v-walk contains a u, v-path), equivalence classes are exactly connected components, Thm (König, characterization of bipartite graphs) G is bipartite iff G does not contain any odd cycle. K"onigsberg bridges problem, drawing the "little house", Definitions: Eulerian trail, Eulerian circuit, Euler's Theorem: G has an Eulerian circuit iff the degree of every vertex is even. Corollary: For $E(G)$ can be partitioned into $\max\{1, k\}$ trails iff the number of odd degree vertices is $2k$. Extremal Problems: Prop: Every n-vertex graph with at most $n-2$ edges is disconnected. (Pf: Lemma: Every graph has at least $v(G) - e(G)$ components), Lemma is best possible: P_n is an n vertex graph with $n-1$ edges that is NOT disconnected, Corollary: minimum number of edges over the family of n-vertex connected graphs is $n-1$; Prop: Every n-vertex graph with $e(G) \geq n$ contains a cycle (Proof: induction on $e(G)$, Lemma: minimum degree $\delta(G) \geq 2$ implies that there is a cycle (Proof: extremality: choose longest path and observe neighbors of endpoints)), Proposition is best possible: P_n is an n vertex graph with $n-1$ edges that contains no cycle. Corollary: maximum number of edges over the family of n-vertex graphs with no cycle is $n-1$. What is the smallest possible lower bound on the minimum degree $\delta(G)$ that would guarantee connectedness? Think of construction? disjoint union of $K_{\lfloor n/2 \rfloor}$ and $K_{\lceil n/2 \rceil}$ has minimum degree $\lfloor n/2 \rfloor - 1$ and is NOT connected. This construction is optimal: Prop: $\delta(G) \geq \lfloor n/2 \rfloor$ implies that G is connected, Corollary: maximum value of δ among n-vertex disconnected graphs is $\lfloor n/2 \rfloor - 1$. What is the largest number of edges on n vertices so there is no triangle? Example on 5 vertices, Construction: $K_{\lfloor n/2 \rfloor}$ is triangle-free and has $\lfloor n^2/4 \rfloor$. This construction is best possible: Thm (Mantel): G does not contain K_3, then $e(G) \leq \lfloor n^2/4 \rfloor$. (Proof: extremality: take a maximum degree vertex and count edges by summing up the degrees in its neighborhood (which is an independent set)).

Week 11: (West 7.2, 1.2, 1.3) Def: Hamilton cycle, Hamiltonian graph. Example: Dodecaeder, Petersen, Special case of TSP, Dirac's Theorem: min-degree $n/2$ guarantees a Hamilton cycle. Precise solution of the extremal problem of the largest minimum degree possible in non-Hamiltonian graphs (with precisely matching construction). Def: acyclic, forest, tree, leaf, spanning subgraph, spanning tree, Examples (paths, stars), Lemma: every tree has at least two leaves, deleting a leaf from a tree produces a tree. Characterization of trees (any two properties of "acyclic", "connected", "has $n-1$ edges" are equivalent with each other; also with "every two vertices has a unique path between them"), Corollaries (every edge of a tree is a cut-edge, adding any new edge to a tree produces exactly one cycle, every connected graph contains a spanning tree) Application (Game of "Bridg-It", proof of existence of a winning strategy of First Player using Strategy Stealing. Explicit winning strategy using Lehman's Theorem. Definition of Maker-Breaker positional game "Connectivity" Proof of Lehman's Theorem using the edge-swapping lemma for spanning trees.

Week 12: (West 3.1) Introduction of matchings as job assignment problem. Definition of matching. Examples in matchings in various graphs. Definition of alternating paths and augmenting paths. Berge's Theorem. Statement and proof. Examples of matchings in bipartite graphs showing intuition behind Hall's condition. Hall's Theorem. Statement and proof. Regular bipartite graphs have a perfect matching. Definition of matching number. Definition of vertex cover number. Proof that cover number ¿= matching number for general graphs. Konig's Theorem. Statement and deduction from Hall's Theorem.

Week 13. (West 4.1) Definition: connectivity of graphs. vertex cut, k-connected, $\kappa(G)$, Examples: $\kappa(K_{n,m}) = \min\{n, m\}$, $\kappa(Q_d) = d$ Definition: edge cut, edge-connectivity of G, $\kappa'(G)$,. k-edge-connected. If G is a simple graph, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$ (*Homework.* Example of a graph G with $\kappa(G) = k$, $\kappa'(G) = l$, $\delta(G) = m$, for any $0 < k \leq l \leq m$.) Characterization of 2-connected graphs (Whitney's Theorem, Let G be a graph, $n(G) \geq 3$. Then G is 2-connected iff for every $u, v \in V(G)$ there exist two internally disjoint u, v-paths in G. Statement of Menger's Theorem.

How many colors are needed to color a map? 4-color problem. (Relevant concepts: colorings, planar graphs). Definition of k-coloring, proper coloring, chromatic number, $\chi(G)$, (Examples: K_n, $K_{n,m}$, C_{2k+1}, Petersen) Def: k-color-critical (Example: 1-, 2-, 3-critical graphs.) Lower bounds ($\chi(G) \geq \omega(G), \chi(G) \geq \frac{n(G)}{\alpha(G)}$) Examples for $\chi(G) \neq \omega(G)$: (odd cycles of length at least 5, complements of odd cycles of order at least 5, random graph $G = G(n, \frac{1}{2})$. Mycielski's Construction. (The bound $\chi(G) \geq \omega(G)$ could be arbitrarily bad.) Thm (If G is triangle-free, then so is $M(G)$, If $\chi(G) = k$, then $\chi(M(G)) = k + 1$.) Upper bounds ($\chi(G) \leq \Delta(G) + 1$, $\chi(G) \leq \max_{H \subseteq G} \delta(H) + 1$. (Algorithmic proof: greedy coloring precedure))

Week 14. (West) Definitions and examples of curves, graph drawings, planar graphs, plane graphs. Statement of the Jordan curve theorem. Proof that K_5 is not planar from the Jordan curve theorem. (non-planarity of $K_{3,3}$ is exercise). Subdivisions. Statement of Kuratowski's Theorem. (Example: Petersen isn't planar). Definitions: faces, duals. Euler's formula. (for connected graphs: by induction on the number of edges; version for more components is stated). Lemma: $e(G) \leq 3n - 6$ for planar graphs ($e(G) \geq 2$) $e(G) \leq 2n - 4$ for triangle-free planara graphs. (Example: K_5 and $K_{3,3}$ are not planar as corollaries.) 6-colour theorem, 5-colour theorem. (Definition of Kempe chain.) 4-colour theorem discussion (fake proof and proof approach)