**Exercise 1**

(a) The Stirling number of the second kind $S_{n,k}$ is the number of partitions of the set $[n]$ into $k$ nonempty subsets (where by a partition we mean a set $\{A_1, \dots, A_k\}$ of pairwise disjoint subsets of $[n]$, such that $\cup_{i=1}^k A_i = [n]$.)

Or, formally,

$$S_{n,k} = \left| \left\{ \{A_1, \dots, A_k\} \in \binom{2^{[n]} \setminus \{\emptyset\}}{k} : \cup_{i=1}^k A_i = [n], \forall i \neq j \ A_i \cap A_j = \emptyset \right\} \right|.$$

(b) Let $n, k \geq 0$. Let $\mathcal{P}_{n+1,k+1}$ be the set of all partitions of $[n+1]$ into $k+1$ nonempty subsets. By definition $|\mathcal{P}_{n+1,k+1}| = S_{n+1,k+1}$. We classify these partitions according to their member which contains the element $n+1$. For any subset $J \subseteq [n]$, let $\mathcal{P}^J_{n+1,k+1} \subseteq \mathcal{P}_{n+1,k+1}$ be the set of partitions which contains $J \cup \{n+1\}$ as a member. Then

$$\mathcal{P}_{n+1,k+1} = \bigcup_{J \in 2^{[n]}} \mathcal{P}^J_{n+1,k+1}$$

is a disjoint union and $|\mathcal{P}^J_{n+1,k+1}|$ is equal to the number of partitions of the set $[n] \setminus J$ into $k$ nonempty parts, which is $S_{n-|J|,k}$. Hence, summing up according to the size of $J$, we get

$$\begin{aligned} S_{n+1,k+1} &= \sum_{J \in 2^{[n]}} |\mathcal{P}^J_{n+1,k+1}| = \sum_{J \in 2^{[n]}} S_{n-|J|,k} \\ &= \sum_{j=0}^{n} \sum_{J \in \binom{[n]}{j}} S_{n-|J|,k} \\ &= \sum_{j=0}^{n} \binom{n}{j} S_{n-j,k} = \sum_{i=0}^{n} \binom{n}{i} S_{i,k}. \end{aligned}$$

In the last step we switched to $i = n - j$ and used that $\binom{n}{n-i} = \binom{n}{i}$.

**Exercise 2**

In order to show that the sequence

$$a_n = \frac{1}{2}((1+\sqrt{2})^n + (1-\sqrt{2})^n)).$$

is integer for all $n$, we check that it is integer for $n = 0$ and $n = 1$ (it is, since $a_0 = 1$ and $a_1 = 1$) and we find a recursion with integer coefficients for $n \geq 2$.

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{1}{2}\left(\sum_{n=0}^{\infty}(1+\sqrt{2})^n x^n + \sum_{n=0}^{\infty}(1-\sqrt{2})^n)x^n\right)$ be the generating function of $a_n$. Then, expanding the geometric series, we have

$$f(x) = \frac{1}{2}\left(\frac{1}{1-(1+\sqrt{2})x} + \frac{1}{1-(1-\sqrt{2})x}\right) = \frac{x-1}{x^2+2x-1}$$

and thus $f(x)(x^2 + 2x - 1) = x - 1$. This means

$$\sum_{n=0}^{\infty} a_n x^n = f(x) = x^2 f(x) + 2x f(x) - x + 1 = \sum_{n=0}^{\infty} a_n x^{n+2} + 2\sum_{n=0}^{\infty} a_n x^{n+1} - x + 1.$$

Comparing the coefficients of $x^n$ on the left hand side and the right hand side of the equality, we obtain that for $n \geq 2$, $a_n = 2a_{n-1} + a_{n-2}$. Since the initial values $a_0 = 1$ and $a_1 = 1$ are integers, the recursion with integer coefficients ensures that $a_n$ is an integer for all $n \geq 0$.

**Exercise 3**

(a) The Ramsey number $R_r(3)$ is the smallest integer $n$ such that for every $r$-coloring of the edges of $K_n$ there exists a monochromatic triangle $K_3$ (that is, a triangle in $K_n$ all edges of which have the same color).

Or, formally,

$$R_r(3) = \min \Big\{ n : \forall \text{ function } c : E(K_n) \to [r] \; \exists \text{ index } i \in [r] \text{ and subset } T \in \binom{V(K_n)}{3} \text{ such that } \forall e \in \binom{T}{2} \; c(e) = i \Big\}$$

(b) We prove the finiteness of $R_{r-1}(3)$ by induction on $r$. To start the induction, we have that $R_1(3) \leq 3$, because coloring the edges of $K_3$ with one color does produce a monochromatic $K_3$.

Let $r \geq 2$. To prove the induction step we show

$$R_r(3) \leq n := r(R_{r-1}(3) - 1) + 2,$$

which implies that $R_r(3)$ is finite, since $R_{r-1}(3)$ is finite by induction. Take an arbitrary $r$-coloring $c : E(K_n) \to [r]$. We need to show that there is a monochromatic triangle.

Let us fix one vertex $v \in V(K_n)$. We classify the neighbours of $v$ by the color of the edge joining them to $v$, i.e. for $i \in [r]$, let

$$N_i(v) = \{w \in V(K_n) \setminus \{v\} : c(vw) = i\}.$$

Hence there must exist an index $j, 1 \leq j \leq r$, such that $|N_j(v)| \geq R_{r-1}(3)$. Indeed, otherwise

$$r(R_{r-1}(3) - 1) \geq \sum_{i=1}^{r} |N_i(v)| = |\bigcup_{i=1}^{r} N_i(v)| = |N_{K_n}(v)| = n - 1,$$

a contradiction to the choice of $n = r(R_{r-1}(3) - 1) + 2$. Now there are two possible cases, each giving us a monochromatic triangle and finishing the proof:

*Case 1:* There exists an edge of color $j$ inside $N_j(v)$. Then this edge together with $v$ gives us a monochromatic triangle of color $j$.

*Case 2:* All edges inside $N_j(v)$ have color different from $j$. Then we have a complete graph on $R_{r-1}(3)$ vertices colored with $r-1$ colors, so by definition of $R_{r-1}(3)$ there is a monochromatic triangle in it.

So $R_r(3) \leq r(R_r(3)-1)+2$ holds, thus $R_r(3)$ is finite and we have finished the proof of the induction step.

**Exercise 4**

- A graph$G$ is called *bipartite* if there exists two independent sets $A, B \subseteq V$ such that $A \cup B = V$ (where a set is called *independent* if it does not contain any edges).

  A graph $G$ with $E(G) = \{e_1, \dots, e_m\}$ is called *Eulerian* if there exists an alternating list $(v_0, e_1, v_1, e_2, \dots e_m, v_m)$ of vertices and edges, such that $e_{i-1} \cap e_i = \{v_i\}$ for every $i = 1, \dots, m-1$, as well as $e_m \cap e_1 = \{v_m\} = \{v_0\}$.

- We prove that the statement is true. Let $G = (V, E)$ be a bipartite Eulerian graph. We proved in the lecture that in an Eulerian graph every vertex has even degree. Furthermore since $G$ is bipartite we can find two disjoint independent sets $A, B$ which partition $V$. Now every edge is incident to exactly one vertex in $A$, so we can enumerate all edges of $G$ exactly once by summing up the degrees of vertices in $A$:

$$\sum_{v \in A} d(v) = |E|.$$

  Hence $|E|$ is a sum of even numbers and therefore is even.

**Exercise 5**

Let $n \geq 3$ and $G = (V, E)$ be a graph with $\delta(G) \geq \frac{n}{2}$. We want to find a Hamiltonian cycle.

Let $(v_1, v_2, \dots, v_k)$ be the vertices of a path $P$ of maximum length in $G$. Both endpoints $v_1$ and $v_k$ have all their neighbours on $P$ because otherwise we could lengthen $P$ by appending such an outside neighbor to it. Since $\delta(G) \geq \frac{n}{2}$ we get that at least $\frac{n}{2}$ of the vertices of $P$ are adjacent to $v_1$ and at least $\frac{n}{2}$ vertices to $v_k$.

We claim that there exists an index $i \in [k-1]$ such that $v_i v_k \in E$ and $v_1 v_{i+1} \in E$. Assume not. Then every neighbour of $v_k$ forbids the next vertex on $P$ to be a neighbour of $v_1$. Therefore the neigbours of $v_1$ have to be among the at most $k-1-d(v_k)$ vertices of $P - v_1$, which do not follow a neighbor of $v_k$. This number is at most $k-1-d(v_k) \leq k-1-\frac{n}{2} < \frac{n}{2}$, contradicting that $d(v_1) \geq \frac{n}{2}$.

With the help of this index $i$, we can find a cycle $(v_1, \dots, v_i, v_k, v_{k-1}, \dots, v_{i+1})$ of length $k$ in $G$. This cycle on $V(P)$ must span a connected component, as any edge leaving it would give us a path of length $k+1$.

But $G$ is connected, as otherwise there was a component of size at most $\frac{n}{2}$ which would give us a minimum degree $\delta(G) \leq \frac{n}{2} - 1$, contradiction.

In conclusion, $G$ is connected and hence $k = n$ and the cycle on $V(P)$ is a Hamilton cycle.

**Exercise 6**

Let $n > 1$ and $d = (d_1, \dots, d_n) \in \mathbb{N}^n$ be a sequence of positive integers. We have to prove that this sequence is the degree sequence of a tree if and only if $\sum_{i=1}^n d_i = 2n - 2$.

"$\Rightarrow$" Let $d = (d_1, \dots, d_n) \in \mathbb{N}^n$ be the degree sequence of a tree $T$. A tree on $n$ vertices has $n - 1$ edges. So if we apply the Handshaking Lemma we get

$$\sum_{i=1}^{n} d_i = 2|E(T)| = 2(n-1) = 2n - 2.$$

"$\Leftarrow$" Let $d = (d_1, \dots, d_n) \in \mathbb{N}^n$ be a sequence of positive integers with $\sum_{i=1}^n d_i = 2n - 2$. We show by induction that there is a tree with this as its degree sequence. For $n = 2$ the only possibility is $(1, 1)$ and this can be realized as the degree sequence of the tree $P_2$.

Let $n > 2$. Assume without loss of generality that $d_1 \geq d_2 \geq \dots \geq d_n$. Note that $d_n = 1$ because otherwise $d_i \geq 2$ for all $i \in [n]$ and therefore $\sum_{i=1}^n d_i \geq 2n > 2n - 2$, which is a contradiction.

Furthermore $d_1 \geq 2$ because otherwise $d_i \leq 1$ for all $i \in [n]$, so $\sum_{i=1}^n d_i \leq n < 2n - 2$ again yields a contradiction for $n > 2$.

We define now a sequence $d'$ of length $n - 1$ as $d'_1 = d_1 - 1$, $d'_i = d_i$ for $i = 2, \dots, n - 1$. This sequence satisfies the condition since by our assumption

$$\sum_{i=1}^{n-1} d'_i = d_1 - 1 + \sum_{i=2}^{n-1} d_i = \left(\sum_{i=1}^{n} d_i\right) - d_n - 1 = (2n-2) - 2 = 2(n-1) - 2.$$

So we can apply the induction hypothesis and get a tree $T'$ on $n - 1$ vertices with $(d'_i)_{i=1}^{n-1}$ as its degree sequence.

Now we add to $T'$ one vertex called $n$ and the edge $\{1, n\}$ to create a graph $T$ on $n$ vertices. The graph $T$ has $n - 1$ edges, is acyclic (we added a pendant vertex to the acyclic graph $T'$) and therefore is a tree. The degree sequence of $T$ is $(d_i)_{i=1}^n$ by construction.