Freie Universität Berlin
Institut für Mathematik
Discrete Mathematics 1

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Exercise sheet 4

Submit only **four(!!!)** exercises by 16th of May, 2PM in the box of Olaf Parczyk

Exercise 1 [10 points]
With all permutations equally likely what is the expected[1] number of cycles?
(*Hint:* Differentiation could come in handy ...)

Exercise 2 [10 points]
How many ways are there to fill completely without overlap an $n \times 2$ rectangle with 1×1 and 2×2 squares?

Exercise 3 [10 points]
Find the generating function of the sequence $(a_0, a_1, \ldots)$ given by the recurrence relation $a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3}$ for every $n \geq 3$ and the initial values $a_0 = 0$, $a_1 = 1$, and $a_2 = 3$ and use it to derive a closed formula for a_n.

Exercise 4 [10 points]
Prove that every integer $n \in \mathbb{N}$ can be written as a sum of mutually distinct Fibonacci numbers.

Exercise 5 [10 points]
Show that $(6 + \sqrt{37})^{999}$ has at least 999 zeros following the decimal point.

Exercise 6 [10 points]
Finish the proof of the theorem in the lecture about homogeneous linear recurrences. Let k be a positive integer and let

$$p(x) = x^k - \alpha_{k-1}x^{k-1} - \cdots - \alpha_1 x - \alpha_0$$

be a polynomial where $\alpha_0, \ldots, \alpha_{k-1} \in \mathbb{C}$. Let $\lambda_1, \ldots, \lambda_q \in \mathbb{C}$ be the distinct roots of $p(x)$, with multiplicity $k_1, \ldots, k_q$, respectively. That is, $k_1 + \cdots k_q = k$ and

$$p(x) = (x - \lambda_1)^{k_1}(x - \lambda_2)^{k_2} \cdots (x - \lambda_q)^{k_q}.$$

[1] In case you have not taken any probability theory yet, here are the necessary definitions for this exercise. For a finite set Ω (which will be the case in our setting) a *random variable* is just a function $X : \Omega \to \mathbb{R}$. A *probability measure* on Ω is given by a function $\mathbb{P} : \Omega \to [0,1]$ with the property that $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$ ($\mathbb{P}(\omega)$ represents that "how likely it is" that the element ω of Ω is selected.) The *expectation of* X (that is, the "average value" of $X(\omega)$) when a random element of Ω is picked according to the probability $\mathbb{P}$ is $\mathbb{E}(X) := \sum_{\omega \in \Omega} \mathbb{P}(\omega)X(\omega)$.
In our case $\Omega = S_n$, $\mathbb{P}$ is the uniform distribution, that is $\mathbb{P}(\omega) = \frac{1}{|\Omega|}$ for every $\omega \in \Omega$, and $X(\omega)$ is the number of cycles in the cyclic representation of permutation ω.

Show that for every sequence $(a_0, a_1, \ldots)$ satisfying the recurrence

$$a_n = \alpha_{k-1} a_{n-1} + \cdots \alpha_0 a_{n-k} \text{ for all } n \geq k$$

there exist constants $C_{ij} \in \mathbb{C}$ for every $i = 1, \ldots q$ and $j = 1, \ldots k_i - 1$, such that for every integer $n \geq 0$ we have

$$a_n = \sum_{i=1}^{q} \sum_{j=0}^{k_i - 1} C_{ij} \binom{n}{j} \lambda_i^n.$$