Freie Universität Berlin
Institut für Mathematik
Discrete Mathematics 1

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Exercise sheet 6

Submit only **four(!!!)** exercises by 30th of May, 2PM in the box of Olaf Parczyk

The function $\pi(n)$ counts the prime numbers up to n, that is,

$$\pi(n) := |\{p \in [n] : \ p \text{ is a prime}\}|^{1}$$

Exercise 1 [10 points]

(a) Show that every prime number p, $m < p \leq 2m$, divides $\binom{2m}{m}$.

(b) Show that $\pi(n) = O\left(\frac{n}{\ln n}\right)$.

Exercise 2 [10 points]

(a) Show that if p^k is a prime power that divides $\binom{2m}{m}$, then $p^k \leq 2m$.

(b) Show that $\pi(n) = \Omega\left(\frac{n}{\ln n}\right)$.

Exercise 3 [10 points]

Prove the formula

$$k!S(n;k) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

(*Hint*: Give a combinatorial meaning to this number and then count two ways.)

Exercise 4 [10 points]

Show the other direction of the Möbius inversion. Let $f, g : \mathbb{N} \to \mathbb{C}$.

$$g(n) = \sum_{\substack{d \in \mathbb{N} \\ d|n}} f(d)\mu(n/d) \ \ \forall n \in \mathbb{N} \quad \Longrightarrow \quad f(n) = \sum_{\substack{d \in \mathbb{N} \\ d|n}} g(d) \ \ \forall n \in \mathbb{N}.$$

[1] The growth rate of $\pi(n)$ fascinated mathematicians for centuries, before finally in 1896 Hadamard and de la Vallée Poussain proved that $\pi(n) \sim \frac{n}{\ln n}$. In the first two exercises you are asked to show a weaker statement, that the *order of magnitude* of $\pi(n)$ is $\frac{n}{\ln n}$. Even though these arguments might seem relatively simple in retrospect, it was only after many decades of unsuccesful tries by such greats as Gauss and Legendre that Chebyshev found a proof in 1852.

Exercise 5 [10 points]

Let d_n be the number of derangements over $[n]$. Prove the formula $n! = \sum_{k=0}^{n} \binom{n}{k} d_k$ and use it to show that the exponential generating function $\hat{D}(z)$ of the the sequence d_k is $\frac{e^{-z}}{1-z}$. (*Hint:* Consider the product of two exponential generating functions.)

Exercise 6 [10 points]

Let us try one more time: finish the proof of the theorem in the lecture about homogeneous linear recurrences. Let k be a positive integer and let

$$p(x) = x^k - \alpha_{k-1}x^{k-1} - \cdots - \alpha_1 x - \alpha_0$$

be a polynomial where $\alpha_0, \ldots, \alpha_{k-1} \in \mathbb{C}$ and $\alpha_0 \neq 0$[2]. Let $\lambda_1, \ldots, \lambda_q \in \mathbb{C}$ be the distinct roots of $p(x)$, with multiplicity $k_1, \ldots, k_q$, respectively. That is, $k_1 + \cdots k_q = k$ and

$$p(x) = (x - \lambda_1)^{k_1}(x - \lambda_2)^{k_2} \cdots (x - \lambda_q)^{k_q}.$$

Show that for every sequence $(a_0, a_1, \ldots)$ satisfying the recurrence

$$a_n = \alpha_{k-1}a_{n-1} + \cdots \alpha_0 a_{n-k} \text{ for all } n \geq k$$

there exist constants $C_{ij} \in \mathbb{C}$ for every $i = 1, \ldots q$ and $j = 0, \ldots k_i - 1$, such that for every integer $n \geq 0$ we have

$$a_n = \sum_{i=1}^{q} \sum_{j=0}^{k_i-1} C_{ij} n^{\underline{j}} \lambda_i^{n-j}.$$

(*Hint:* Note that a root λ of a polynomial has multiplicity at least 2 if and only if λ is also root of the derivative.)

[2] Note that $\alpha_0 \neq 0$ is not a real restriction: the coefficient of the last term a_{n-k} of the recurrence can always be assumed to be non-zero, otherwise we have a recurrence with fewer terms.