Exercise 1

Kuratowski's Theorem says that a graph is planar if and only if it does not contain a subdivision of K_5 and does not contain a subdivision of $K_{3,3}$.

Let G be an outerplanar graph. Let us construct a supergraph $G' \supseteq G$ by adding a new vertex v and connecting it to every vertex of G. We claim that G' is planar. For this, fix first an embedding of G such that all vertices are on the boundary of the outer face. Then we can add the new vertex v in the outer face and connect it to all vertices of G without any crossing, giving a planar embedding of G'. Hence G' is planar and thus does not contain a subdivision of K_5 or $K_{3,3}$ by Kuratowski's Theorem. If G contained a subdivision of K_4 or $K_{2,3}$, then adding v and the edges from v to the branch vertices of this subdivision would produce a subdivision of K_5 or $K_{3,3}$ in G', a contradiction. Hence G contains neither a K_5-subdivision nor a $K_{2,3}$-subdivision.

For the reverse implication let G be a graph not containing a subdivision of K_4 or $K_{2,3}$. Again we construct G' by adding a vertex v and connecting it to all vertices of G. Then G' does not contain a subdivision of K_5 or $K_{3,3}$, so it is planar by Kuratowski's Theorem and we can consider an embedding of G' without crossing of its edges. If v does not lie on the boundary of the outer face, we can apply a map onto the sphere and back onto the plane to make one of the faces next to v the outer face. If we remove v then all vertices are on the boundary of the outer face, because they were connected to v in an embedding without intersections of edges. Thus G is outerplanar.

Exercise 2

First we prove that every simple outerplanar graph H on $n \geq 2$ vertices has a vertex of degree 2. Let us add edges to H preserving outerplanarity until it is possible, that is, when we have a simple supergraph $G \supseteq H$ with $V(G) = V(H) = V$, such that for every $e \in \binom{V}{2} \setminus E(G)$ the graph $G + e$ is not outerplanar. If we show that the minimum degree of G is at most 2, then the same is also true for H, as $d_H(v) \leq d_G(v)$ for all vertex $v \in V$.

We claim that G is 2-connected. Suppose not. Let us take an outerplanar embedding of G. If G were not connected then we could draw a non-crossing curve between any two vertices u and v in different components of G and still every vertex of G would be on the boundary of the infinite face. Hence the graph $G + uv$ would be outerplanar, contradicting our assumption about G. If G had connectivity 1, then let v be a cut-vertex and let C_1 and C_2 be two components in $G - v$. Let $u_1 \in C_1$ and $u_2 \in C_2$ be neighbors of v, such that the drawing of the edges vu_1 and vu_2 are leaving v right after each other when we go around v in a very small circle. Two such neighbors certainly exists as v has neighbors both in C_1 and C_2. Now it is possible to draw a non-crossing curve between u_1 and u_2 by following closely first the drawing of the edge u_1v and then the drawing of the edge vu_2. This curve will close a triangle with the curves u_1v and vu_2 with no vertex in its interior, so the obtained drawing of $G + u_1u_2$ is outerplanar, contradicting our assumption on G.

Hence G is 2-connected. Take a cycle C of maximum length in G. We claim that C is a Hamilton cycle. If not, then there is a vertex $u \in V(G) \setminus V(C)$.

Let us take an arbitrary edge $e \in E(C)$ on the cycle and an arbitrary edge f incident to u. By our characterization of 2-connected graphs, there exists a cycle R in G containing both e and f. Let v_1 and v_2 be the vertices of R closest to u on R, in the two different directions. In other words: the v_1, v_2-arc R' of R containing u does not contain any other vertices on C, but v_1 and v_2. By the Jordan Curve Theorem the arc R' separates the outside region of C into two further regions, exactly one of them finite. Let us consider now the cycle C' which we obtain from C by replacing the v_1v_2-arc of C bounding the *finite* face of this separation, with the arc R'. Now if v_1 and v_2 are neighboring on C, then C' would contain all vertices of C and at least one more, vertex u, hence C' would contradict the maximality of C. However, if v_1 and v_2 are not adjacent, then any vertex between them on C would be on the inside of the cycle C' and would not be on the boundary of the infinite face of the outerplanar embedding of G, also a contradiction.

So C is a Hamilton cycle and hence all further edges of G must go in the interior of C (otherwise a vertex of C would not be on the boundary of the infinite face by the Jordan Curve Theorem). If there are no interior edges then $C = G$ and every vertex has degree 2. Otherwise let us take an interior edge yz such that the distance of y and z on C is minimum (if there are more than one pairs with the same distance, then take one such arbitrarily). The distance of y and z on C is of course at least 2, because G is simple, so multiple edges are not allowed. But then, any vertex between y and z on a shortest y, z-path on C must have degree 2. (otherwise any interior edge emanating from such a vertex would have endpoints that are closer to each other on C than y and z, a contradcition).

So we proved that every outerplanar graph has minimum degree at most 2.

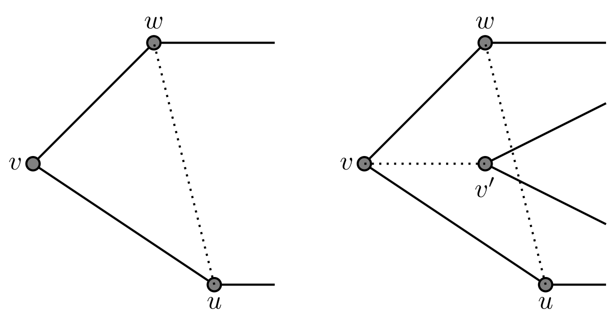
Next we want to show that every outerplanar graph is 2-degenerate and then use that we proved in the lecture that every d-degenerate graph is $d+1$-colorable (A proof of this with induction would also not be difficult. Try it!), so every outerplanar graph is $2 + 1 = 3$-colorable.

So, recall that 2-degeneracy means that every subgraph of G has a vertex of degree at most 2. Since we have just shown that every outerplanar graph has a vertex of degree at most 2, it would be enough to see that any subgraph $G' \subseteq G$ of an outerplanar graph G is outerplanar. For this, take an outerplanar embedding of G and consider its restriction to the vertices and edges of G'. This is an outerplanar embedding of G', since the infinite face did not get smaller, hence all vertices that were on the boundary of the infinite face before, stay on its boundary after the deletion of edges and vertices.

Exercise 3

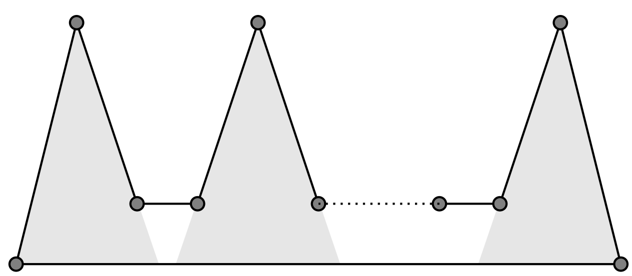
Take a simple polygon P (i.e., without holes) with $n \geq 3$ sides and vertices. We consider P is a plane graph and triangulate it, that is we iteratively add non-crossing diagonals to the drawing until every finite face of the embedding is a triangle. (We say that a line segment is a *diagonal* if it connects two vertices of P and is fully contained inside P.)

First we prove that there always exists a triangulation. For $n = 3$ we only have one triangle. For $n > 3$ we just need to find one diagonal, draw it, hence separating our polygon into two smaller ones which we can triangulate by induction. Let v be the leftmost (according to x-coordinate) vertex of P and u,w its two neighbours. If the line segment form u to w is a diagonal, we are done. Otherwise let v' be the leftmost vertex inside the triangle uvw, then the line segment from v to v' is a diagonal and we are also done.



Now we can view the triangulated polygon P as an embedding of a graph G into the plane with all vertices on the boundary of the outer face, i.e. G is outerplanar. By Exercise 2 we know that G is 3-colourable. Every internal face of the embedding is a triangle, so it has a vertex of every colour. Thus every colour class is a valid set of guards: any (interior or boundary) point of P lies in one of the faces of the triangulation, this face is a triangle and hence any vertex of it "sees" every point in it. Hence the smallest colour class is a valid set of guards of size $\lfloor n/3 \rfloor$.

The following figure is an example for a polygon achieving this bound. (The upper vertices can only be seen by guards that are placed in the triangle "under them". If the upper angles are small enough and the triangles are high enough then these triangles are pairwise disjoint, hence one guard must be in each one of them.)



Exercise 4

Let H be a simple planar graph with $n \geq 4$ vertices. We iteratively add edges to H which preserve its planarity and at the end we obtain a simple supergraph $G \supseteq H$ such that for all $e \in \binom{V(G)}{2} \setminus E(G)$, the graph $G + e$ is not planar. If G

has at most four vertices with degree less than 6, then this is also true for its subgraph H.

We claim that every vertex in G has degree at least 3. Consider a planar embedding of G, such that a minimum degree vertex v is on the boundary of the infinite face. We will show that if $d_G(v) \leq 2$, then there is a face F with v on its boundary, such that v has a non-neighbor $u \neq v$ on the boundary of F. Then we could immediately add a non-crossing curve from v to u within the face F, hence proving that the graph $G + e$ with $e = uv \in \binom{V(G)}{2} \setminus E(G)$ is planar, a contradiction.

The infinite face always has at least three vertices on its boundary (note that $n \geq 4$). Hence if $d_G(v) = 0$ or 1, then v has a non-neighbor on the infinite face. This is also the case if $d_G(v) = 2$ and the infinite face has at least four vertices. If $d_G(v) = 2$ and the infinite face contains exactly three vertices, then all the remaining $n - 3 \geq 1$ vertices are in the interior of the boundary of the infinite face. In particular, the boundary of the infinite face is a triangle and v is adjacent to both other vertices of the infinite face. Since there is at least one more vertex in the interior of this triangle, the finite face which has v on its boundary must have at least four vertices and hence there is a non-neighbor of v on it.

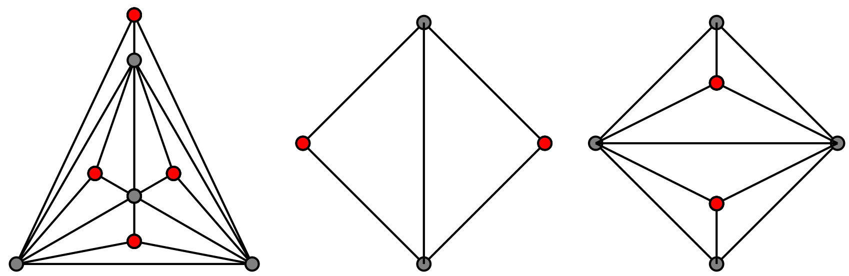
So we have shown that the minimum degree of G is at least three. Let us assume that we have k vertices with degree less that 6 and $k \leq 3$. These vertices have degree at least 3 and the remaining vertices have of course degree at least 6, thus

$$2e(G) = \sum_{v \in V} d(v) = \sum_{\substack{v \in V \\ d(v) < 6}} d(v) + \sum_{\substack{v \in V \\ d(v) \geq 6}} d(v) \geq 3k + 6(n - k) = 6n - 3k \geq 6n - 9.$$

So $e(G) \geq 3n - 4.5$. This is a contradiction to the fact that G is planar, since we know from the Corollary of Euler's Theorem that in a planar graph with $n \geq 3$ the number of edges is at most $3n - 6$. Hence G has at least four vertices of degree less than 6.

For the construction see the figures below. On the left there is the case of $n = 8$ vertices. The four red vertices depict those of degree 3, all other have degree 6. To construct a graph on $n + 2$ vertices from n vertices, we start with any two neighboring triangle face such that two red vertices are non-adjacent (see the middle figure). We remove the diagonal and replace it with a diagonal between the two red vertices. Then we add two vertices in the middle of the two triangle faces and connect them to the three vertices of their respective triangles. This way the four "old" vertices all have degree 6 and the two new ones have degree 3. (see the right figure)

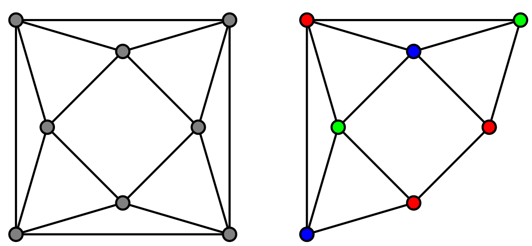
Note that this way we can always add two vertices keeping the number of red vertices four, because the structure shown in the middle always reconstructs.



Exercise 5

Since G_n is planar we know by the Four Color Theorem that there exists a proper coloring of G_n with four colors. If we look at any two consecutive 4-cycles in G_n they form a subgraph which is isomorphic to G_2. So it suffices to prove the statement for G_2. Let us consider a proper 4-coloring of G_2. If there was a color i that appears only once, then after the deletion of the vertex v with color i, the graph $G - v$ would be properly 3-colored, but that is not possible (To check this one colors first a triangle with red, green and blue and then the colors of all other vertices are forced if we shoot for a proper 3-coloring of $G-v$, and at the end an edge with identically colored endpoints is forced; see picture (G_2 is vertex transitive, so it is enough to verify this with the deletion of one arbitrary vertex)).

So every color appears at least two times. Since there are only eight vertices, each of the four colors must appear exactly two times.



Exercise 6

Crossing-free drawings for K_6 on sphere and K_7 on torus.