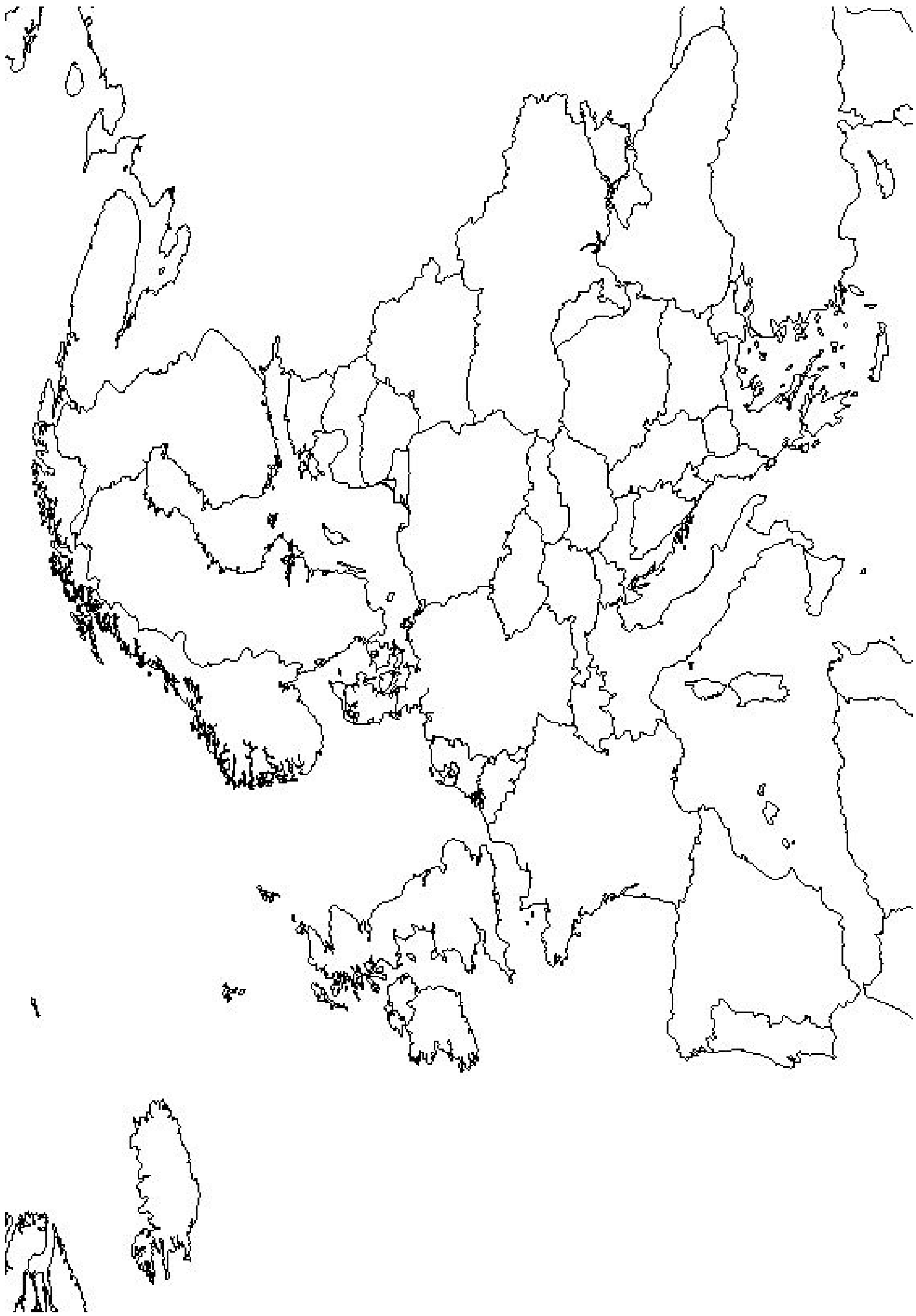
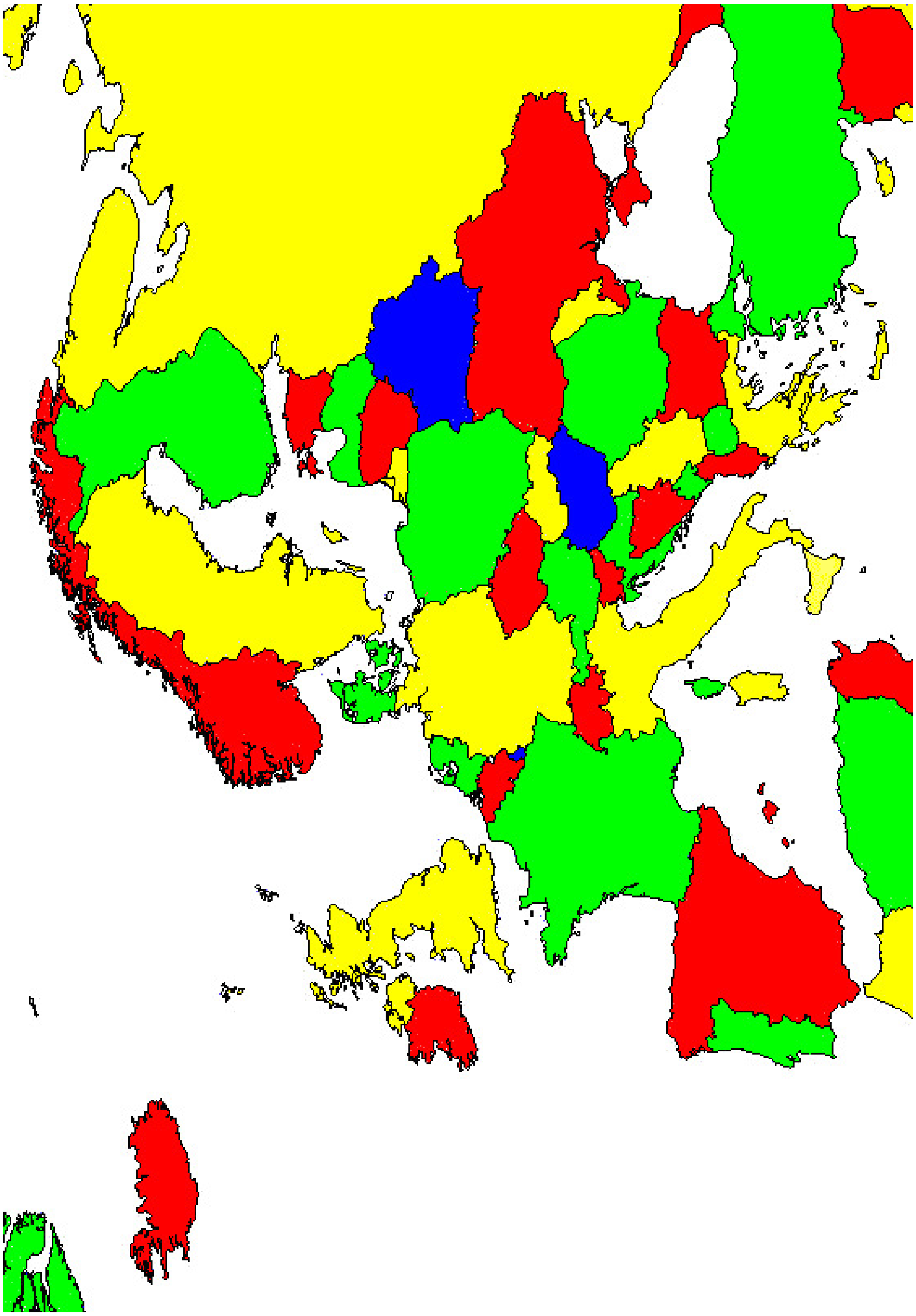
# How many colors are needed to color a map?



# Is 4 always enough?



## Two relevant concepts

How many colors do we need to color a map so neighboring countries get different colors?
Simplifying assumption (not true in reality): Each country is bounded by a simple continuous curve.

**Auxiliary graph:** $V(G) =$ set of countries, $E(G) =$ pairs of countries that are neighboring (share a 1-dimensional piece of their boundary. (just points are not enough!)

**Graph colorings:** We then want a coloring of the *vertices* of this auxiliary graph, such that adjacent vertices receive distinct colors.

**Planar graphs:** The auxiliary graph $G$ of the map has a special property: it can be drawn into the plane such that the edges do not cross. Indeed: draw the vertex representing the country in the “middle” (the “capitol”) and draw a curve to the middle of the boundary curve with each country. This drawing forms an embedding of the graph $G$ in the plane so that the edges do not intersect.

## Vertex coloring, chromatic number

A $k$-coloring of a graph $G$ is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are called colors; the vertices of one color form a color class.

A $k$-coloring is proper if adjacent vertices have different labels. A graph is $k$-colorable if it has a proper $k$-coloring.

The chromatic number is

$$\chi(G) := \min\{k : G \text{ is } k\text{-colorable}\}.$$

A graph $G$ is $k$-chromatic if $\chi(G) = k$.

*Examples.* $K_n$, $K_{n,m}$, $C_5$, Petersen

A graph $G$ is $k$-color-critical (or $k$-critical) if $\chi(H) < \chi(G) = k$ for every *proper* subgraph $H$ of $G$.

*Characterization* of 1-, 2-, 3-critical graphs.

# Lower bounds

## Simple lower bounds

$$\begin{aligned} \chi(G) &\geq \omega(G) \\ \chi(G) &\geq \frac{v(G)}{\alpha(G)} \end{aligned}$$

*Examples* for $\chi(G) \neq \omega(G)$:

- odd cycles of length at least 5,

$$\chi(C_{2k+1}) \geq \frac{v(C_{2k+1})}{\alpha(C_{2k+1})} = 2+\frac{1}{k} > 2 = \omega(C_{2k+1})$$

- complements of odd cycles of order at least 5,

$$\chi(\overline{C}_{2k+1}) \geq \frac{v(\overline{C}_{2k+1})}{\alpha(\overline{C}_{2k+1})} = k+\frac{1}{2} > k = \omega(\overline{C}_{2k+1})$$

- random graph $G = G(n, \frac{1}{2})$, almost surely

$$\chi(G) \approx \frac{n}{2 \log n} > 2 \log n \approx \omega(G)$$

## Mycielski's Construction

The bound $\chi(G) \geq \omega(G)$ could be arbitrarily bad.

**Construction.** Given graph $G$ with vertices $v_1, \ldots, v_n$, we define supergraph $M(G)$.

$V(M(G)) = V(G) \cup \{u_1, \ldots u_n, w\}$.

$E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}\}$.

**Theorem.**

$(i)$ If $G$ is triangle-free, then so is $M(G)$.

$(ii)$ If $\chi(G) = k$, then $\chi(M(G)) = k + 1$.

**Upper bounds** $\chi(G) \leq \Delta(G) + 1$.

*Proof.* Algorithmic. Greedy coloring.

## Jordan Curves

A curve is a subset of $\mathbb{R}^2$ of the form

$$\alpha = \{\gamma(x) : x \in [0,1]\} ,$$

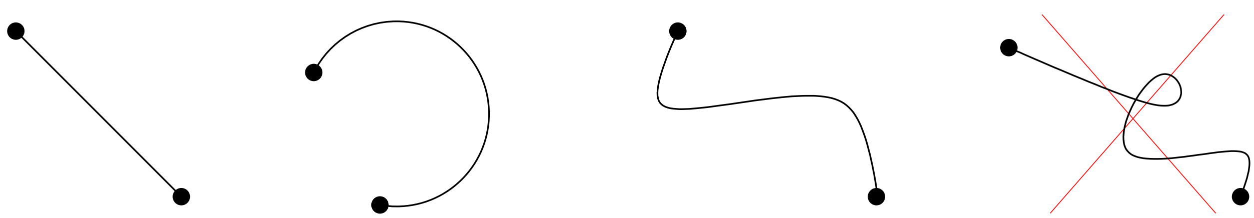
where $\gamma : [0,1] \to \mathbb{R}^2$ is a continuous mapping from the closed interval $[0,1]$ to the plane. $\gamma(0)$ and $\gamma(1)$ are called the *endpoints* of curve $\alpha$.

A curve is closed if its first and last points are the same. A curve is simple if it has no repeated points except possibly first = last. A closed simple curve is called a Jordan-curve.

Examples: Line segments between $p, q \in \mathbb{R}^2$

$$x \mapsto xp + (1-x)q ,$$

circular arcs, Bezier-curves without self-intersection, etc...



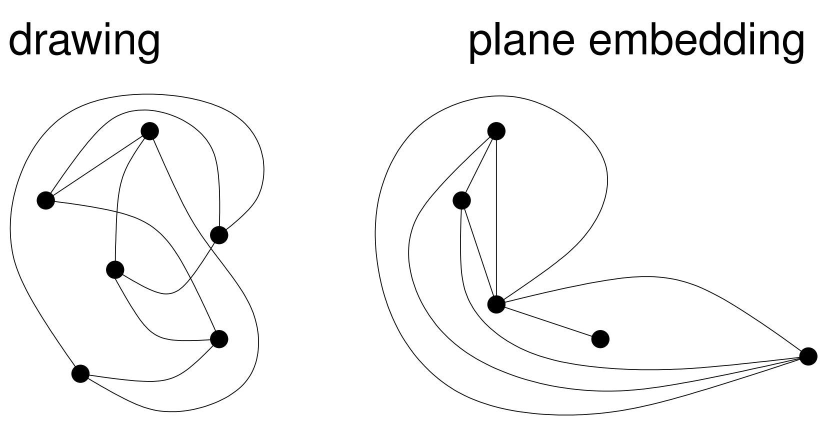
## Drawing of graphs

A drawing of a multigraph $G$ is a function $f$ defined on $V(G) \cup E(G)$ that assigns

- a point $f(v) \in \mathbb{R}^2$ to each vertex $v$ and
- an $f(u), f(v)$-curve to each edge $uv$,

such that the images of vertices are distinct. A point in $f(e) \cap f(e')$ that is not a common endpoint is a crossing.

A multigraph is planar if it has a drawing without crossings. Such a drawing is a planar embedding of $G$. A planar (multi)graph *together* with a particular planar embedding is called a plane (multi)graph.



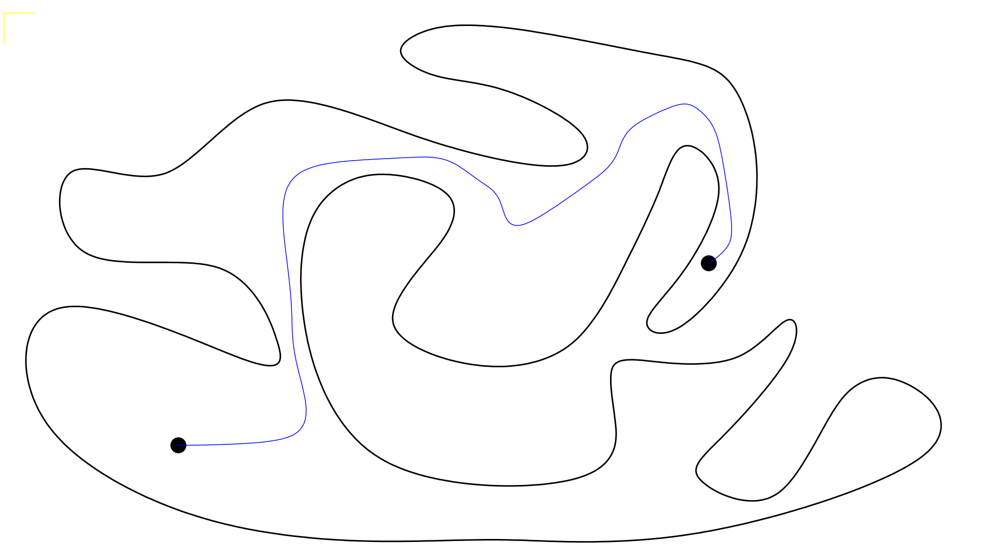
## Are there non-planar graphs?

**Proposition.** $K_5$ and $K_{3,3}$ cannot be drawn without crossing.

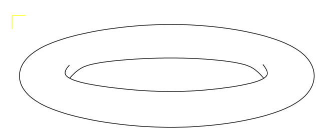
*Proof.* Define the *conflict graph* of edges.

**The unconscious ingredient.**

**Jordan Curve Theorem.** A simple closed curve $C$ partitions the plane into exactly two faces, each having $C$ as boundary.



Not true on the torus!



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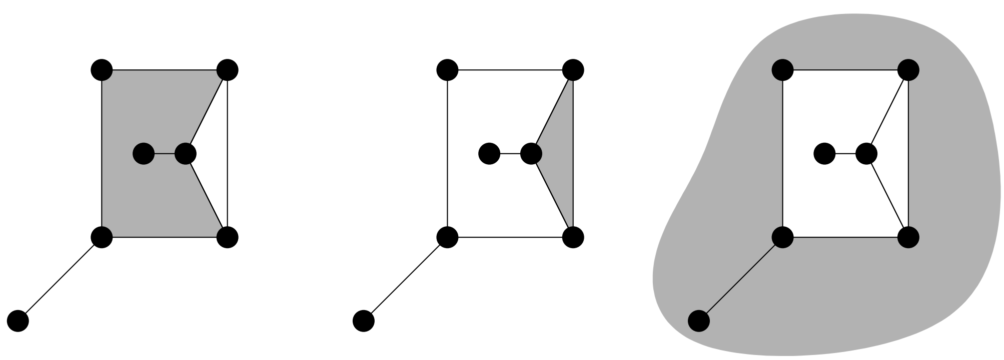
## Regions and faces

An open set in the plane is a set $U \subseteq R^2$ such that for every $p \in U$, all points within some small distance belong to $U$. A region is an open set $U$ that contains a $u, v$-curve for every pair $u, v \in U$. The faces of a plane multigraph are the maximal regions of the plane that contain no points used in the embedding.

A finite plane multigraph $G$ has one unbounded face (also called outer face).



## Dual graph

Denote the set of faces of a plane multigraph $G$ by $F(G)$ and let $E(G) = \{e_1, \ldots, e_m\}$. Define the dual multigraph $G^*$ of $G$ by

- $V(G^*) := F(G)$
- $E(G^*) := \{e_1^*, \ldots, e_m^*\}$, where the endpoints of $e_i^*$ are the two (not necessarily distinct) faces $f', f'' \in F(G)$ on the two sides of $e_i$.

**Remarks.** Multiple edges and/or loops *could* appear in the dual of simple graphs

Different planar embeddings of the *same* planar graph could produce *different* duals.

**Proposition.** Let $l(F_i)$ denote the length of face $F_i$ in a plane multigraph $G$. Then

$$2e(G) = \sum l(F_i).$$

## Euler's Formula

**Theorem.**(Euler, 1758) If a plane multigraph $G$ with $k$ components has $n$ vertices, $e$ edges, and $f$ faces, then

$$n - e + f = 1 + k.$$

*Proof.* Induction on $e$.

*Base Case.* If $e = 0$, then $n = k$ and $f = 1$.

Suppose now $e > 0$.

*Case 1.* $G$ has a cycle.

Delete one edge from a cycle. In the new graph:

$e' = e - 1$, $n' = n$, $f' = f - 1$ (Jordan!), and $k' = k$.

*Case 2.* $G$ is a forest.

Delete a pendant edge. In the new graph:

$e' = e - 1$, $n' = n$, $f' = f$, and $k' = k + 1$.

**Remark.** The dual may depend on the embedding of the graph, but the number of faces does *not*.

## When is a graph planar?

**Corollary** If $G$ is a simple, planar graph with $n(G) \geq 3$, then $e(G) \leq 3n(G) - 6$.
If also $G$ is triangle-free, then $e(G) \leq 2n(G) - 4$.

**Corollary** $K_5$ and $K_{3,3}$ are non-planar.

The subdivision of edge $e = xy$ is the replacment of $e$ with a new vertex $z$ and two new edges $xz$ and $zy$. The graph $H'$ is a subdivision of $H$, if one can obtain $H'$ from $H$ by a series of edge subdivisions. Vertices of $H'$ with degree at least three are called branch vertices.

**Theorem** (Kuratowski, 1930) A graph $G$ is planar iff $G$ does not contain a subdivision of $K_5$ or $K_{3,3}$.

## Coloring maps with 5 colors

**Six Color Theorem.** If $G$ is planar, then $\chi(G) \leq 5$.

*Proof.* By Euler, minimum degree is at most 5. Then

**Proposition** $\chi(G) \leq \max_{H \subseteq G} \delta(H) + 1$.

*Proof.* Greedy coloring procedure with the ordering $v_1, \ldots, v_n$, where $v_i$ is a min-degree vertex of the graph $G[\{v_1, \ldots, v_n\}]$.

**Five Color Theorem.** (Heawood, 1890) If $G$ is planar, then $\chi(G) \leq 5$.

*Proof.* Take a minimal counterexample.

$(i)$ There is a vertex $v$ of degree at most 5.

$(ii)$ Modify a proper 5-coloring of $G - v$ to obtain a proper 5-coloring of $G$. A contradiction.
(*Idea of modification:* Kempe chains.)

# Coloring maps with 4 colors

**Four Color Theorem.** (Appel-Haken, 1976) For any planar graph $G$, $\chi(G) \leq 4$.

*Idea of the proof.*
W.l.o.g. we can assume $G$ is a planar triangulation.
A configuration in a planar triangulation is a separating cycle $C$ (the ring) together with the portion of the graph inside $C$.
For the Four Color Problem, a set of configurations is an unavoidable set if a minimum counterexample must contain a member of it.
A configuration is reducible if a planar graph containing it cannot be a minimal counterexample.

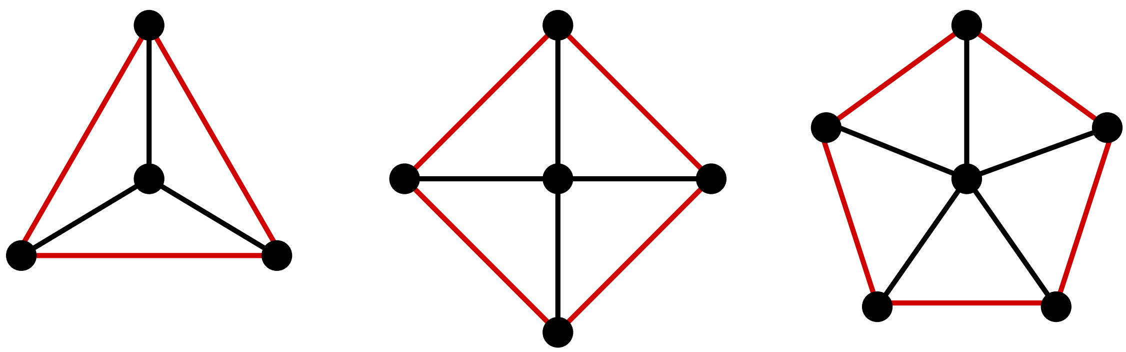
The usual proof attempts to

$(i)$ find a set $\mathcal{C}$ of unavoidable configurations, and

$(ii)$ show that each configuration in $\mathcal{C}$ is reducible.

## Proof attempts of the Four Color Theorem

Kempe's original proof tried to show that the unavoidable set



is reducible.

Appel and Haken found an unavoidable set of 1936 of configurations, (all with ring size at most 14) and proved each of them is reducible. (1000 hours of computer time)

Robertson, Sanders, Seymour and Thomas (1996) used an unavoidable set of 633 configuration. They used 32 rules to prove that each of them is reducible. (3 hours computer time)