Asymptotic Counting, asymptotic Combinatorics

Suppose we obtain the following formula for a sequence

$$a_n = \frac{1}{2}(3-\sqrt{6})^n + \frac{7}{5}(3+\sqrt{6})^n + 101\cdot 2^n$$

Great to know precisely, but:

How large is a_n, really?

$$(3-\sqrt{6}) < 1 \quad \Longrightarrow \quad (3-\sqrt{6})^n \longrightarrow 0$$

How about $\frac{7}{5}\cdot(3+\sqrt{6})^n$ and $101\cdot 2^n$?

Important: $3+\sqrt{6} > 2$

$$\Longrightarrow \frac{\frac{7}{5}\cdot(3+\sqrt{6})^n}{101\cdot 2^n} = \frac{7}{505}\cdot\underbrace{\left(\frac{3}{2}+\frac{\sqrt{6}}{2}\right)}_{>1}{}^{n} \longrightarrow \infty$$

That is: $\frac{7}{5}(3+\sqrt{6})^n$ is eventually larger than $10^{10^{10^{10^{10^{10}}}}}\cdot(101)\cdot 2^n$

So for large enough n we hardly see the little modification $101\cdot 2^n$ causes to $\frac{7}{5}(3+\sqrt{6})^n$ in the formula for a_n

Asymptotic Notation

$f, g : \mathbb{N} \to \mathbb{R}$

We write.
- $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ "g grows much much faster than f"

- $f(n) = O(g(n))$ if $\exists n_0 \in \mathbb{N}, \exists C \in \mathbb{R}$ s.t.
  $\forall n \geq n_0 \quad |f(n)| \leq C|g(n)|$

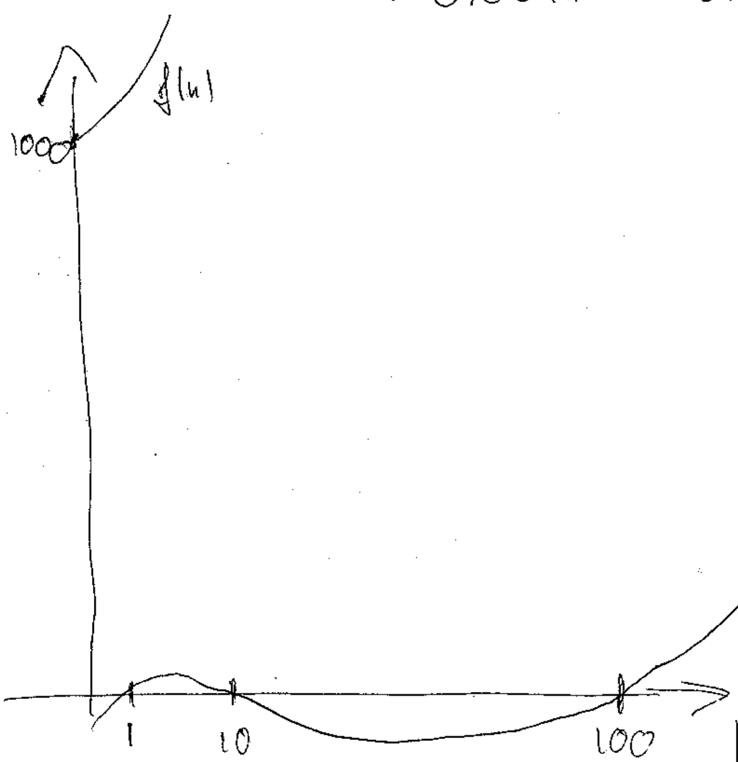
  "f grows at most as fast as g"

Example: $f(n) = 115n^3 + 1000$

$g(n) = 0.001(n-1)(n-10)(n-100) =$

$= 0.001n^3 - 0.001 \cdot 111 \cdot n^2 + 0.001 \cdot 1110n - 1000 \cdot 0.001$

$= 0.001n^3 - 0.111n^2 + 1.11n - 1$



$f(n) \neq o(g(n))$

$g(n) \neq o(f(n))$, because

$\lim \frac{f(n)}{g(n)} = \frac{115}{0.001} = 115000 \neq \infty, \neq 0$

But: $f(n) = O(g(n))$

and $g(n) = O(f(n))$

$|f(n)| \leq 115n^3 + 1000 \leq 116n^3$ for $\forall n \geq 10$

$\leq 1000000\, g(n)$ for $\forall n \geq$

$|g(n)| \underset{\forall n \geq 1}{\leq} 1.111n^3 \leq 115n^3 \leq f(n)$

$$\left.\begin{array}{l} f_1(n) = O(g_1(n)) \\ f_2(n) = O(g_2(n)) \end{array}\right\} \Longrightarrow \begin{array}{l} f_1(n) + f_2(n) = O(g_1(n) + g_2(n)) \\ f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n)) \end{array}$$

$$\left.\begin{array}{l} f_1(n) = o(g_1(n)) \\ f_2(n) = o(g_2(n)) \end{array}\right\} \Longrightarrow \begin{array}{l} f_1(n) + f_2(n) = o(\max\{g_1(n), g_2(n)\}) \\ f_1(n) \cdot f_2(n) = o(g_1(n) \cdot g_2(n)) \end{array}$$

Notation simplifies counting, statement of results, allows us to concentrate on what's important.

$$(101n^2 - 57n + 90)(n^2 - 55n - 101) = O(n^2) \cdot O(n^2) = \boxed{O(n^4)}$$

- $n^\alpha = o(n^\beta) \quad \forall\ \beta > \alpha > 0$
- $n^c = o(a^n) \quad \forall a > 1 \quad \forall c > 0$
- $(\ln n)^c = o(n^\alpha) \quad \forall c > 0 \quad \forall \alpha > 0$

Further Notation

$f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$

$f(n) = \Theta(g(n))$ if $g(n) = O(f(n))$ and $f(n) = O(g(n))$

$f(n) \approx g(n)$ if $\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = 1$

Returning to our original sequence $a_n = \frac{1}{2}(3-\sqrt{6})^n + \frac{7}{5}(3+\sqrt{6})^n + 101 \cdot 2^n$

- $a_n = \Theta((3+\sqrt{6})^n)$
- $a_n \approx \frac{7}{5} \cdot (3+\sqrt{6})^n$
- $a_n = \frac{7}{5}(3+\sqrt{6})^n + O(2^n)$

From rough estimates to more and more precise

Example: Sum of cubes

$$\sum_{i=1}^{n} i^3 \le \sum_{i=1}^{n} n^3 = n^4$$

$$\sum_{i=1}^{n} i^3 \ge \sum_{i=\frac{n}{2}}^{n} i^3 \ge \sum_{i=\frac{n}{2}}^{n} \left(\frac{n}{2}\right)^3 = \left(\frac{n}{2}\right)^4 = \frac{n^4}{16}$$

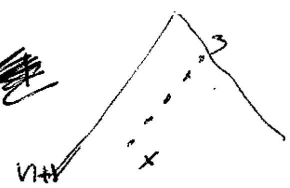
$$\longrightarrow \sum_{i=1}^{n} i^3 = \Theta(n^4)$$

More Precise:

Idea: $\binom{i}{3}$ is also a cubic polynom (as i^3)

but we know a formula for the sum of values

upto n $\sum_{i=0}^{n} \binom{i}{3} = \binom{n+1}{4}$



$$\sum_{i=1}^{n} i^3 = 6\sum_{i=0}^{n} \binom{i}{3} + \sum_{i=0}^{n} \Big[i^3 - 6\underbrace{\binom{i}{3}}_{=\,i(i-1)(i-2)} \Big]$$

$$= 6\binom{n+1}{4} + \sum_{i=0}^{n} -3i^2 + 2i$$

$$= 6\,\frac{(n+1)(n)(n-1)(n-2)}{4!} - 3\sum_{i=0}^{n} i^2 + 2\sum_{i=0}^{n} i$$

$$= \frac{n^4}{4} + O(n^3) \approx \frac{n^4}{4}$$

Example: The Factorial (Of course, there is Stirling's Formula...)

First:

$$n! = \prod_{i=1}^{n} i \le \prod_{i=1}^{n} n = \boxed{n^n}$$

$$n! = n\cdot(n-1)\cdot\ldots\cdot 3\cdot 2\cdot 1 \ge \underbrace{2\cdot\ldots\cdot 2}_{n-1} = \boxed{2^{n-1}}$$

Is the upper or the lower bound closer to the truth?

Second:

$$n! = \left(\prod_{i=1}^{n/2} i\right)\cdot\left(\prod_{i=\frac{n}{2}+1}^{n} i\right) \le \left(\frac{n}{2}\right)^{\frac{n}{2}}\cdot n^{\frac{n}{2}} = \boxed{\frac{n^n}{(\sqrt{2})^n}}$$

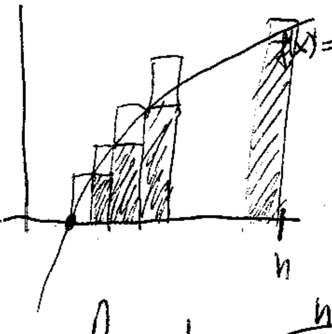
$$n! \ge \prod_{i=\frac{n}{2}+1}^{n} i \ge \prod_{i=\frac{n}{2}+1}^{n} \frac{n}{2} = \boxed{\left(\frac{n}{2}\right)^{\frac{n}{2}}} \quad \left(\gg 10000^n\right)$$

Lower bound is significantly better than before

- LARGER THAN ANY exponential function
- ROUGHLY SQUAREROOT OF UPPER BOUND

Third:

$$\frac{n^n}{e^n} \le n! \le \frac{n^n}{e^n}(n+1)$$



$$\int_1^n \ln x\,dx = \Big[x\ln x - x\Big]_1^n = n\ln n - n$$

$$\ln n! = \sum_{i=1}^{n} \ln i \le \int_1^{n+1} \ln x\,dx = (n+1)\ln(n+1) - (n+1) \Rightarrow n! \le \frac{(n+1)^{n+1}}{e^{n+1}} = \frac{n^n}{e^n}\left(1+\frac{1}{n}\right)^n\cdot\frac{n+1}{e}$$

$$\ge \int_1^n \ln x\,dx = n\ln n - n \Rightarrow n! \ge \frac{n^n}{e^n}$$

Fourth: Stirlings Formula $n! \approx \frac{n^n}{e^n}\cdot\sqrt{n}\cdot\sqrt{2\pi}$

Example: Estimating binomial coefficients $\binom{n}{k}$ when $n \to \infty$

True for ALL $n \geq k \geq 0$

$$\frac{n^k}{k^k} \leq \binom{n}{k} \leq \frac{n^k}{k!}$$

$$\frac{n(n-1)\cdots(n-k+1)}{k!} \leq \frac{n \cdot n \cdots n}{k!}$$

$$\frac{n}{k}\cdot\frac{(n-1)}{k-1}\cdots\frac{(n-k+1)}{1} \geq \underbrace{\frac{n}{k}\cdot\frac{n}{k}\cdots\frac{n}{k}}_{k\text{-times}}$$

Asymptotics:

Case 1. k is constant $\binom{n}{k} \approx \frac{n^k}{k!}$ (Upper bound is asymptotically tight)

Upper bound for all $n \geq k \geq 0$

Thm:

$$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$$

Pf: Binomial Thm:

$$(1+x)^n = \sum_{i=0}^{n}\binom{n}{i}x^i \geq \sum_{i=0}^{k}\binom{n}{i}x^i \quad / : x^k$$

$$\left(\frac{1+x}{x^k}\right)^n \geq \sum_{i=0}^{k}\binom{n}{i}\frac{1}{x^{k-i}} \geq \sum_{i=0}^{k}\binom{n}{i}$$

$\forall x \in (0,1)$

Choose $x = \frac{k}{n}$ and substitute

WHY THIS?

Differentiate

$$n(1+x)^{n-1} \cdot x^k - (1+x)^n k \cdot x^{k-1} = 0$$

$$nx - (1+x)k = 0$$

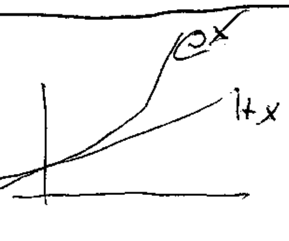
$$x(n-k) = k$$

$$x = \frac{k}{n-k}$$

$$\left(\frac{en}{k}\right)^k = \frac{\left(e^{\frac{k}{n}}\right)^n}{\left(\frac{k}{n}\right)^k} \geq \left(\frac{1+\frac{k}{n}}{\left(\frac{k}{n}\right)^k}\right)^n \geq \sum_{i=0}^{k}\binom{n}{i}$$

Prop $\forall x \in \mathbb{R} \quad 1+x \leq e^x$

Pf: $1+x$ is the tangent of e^x

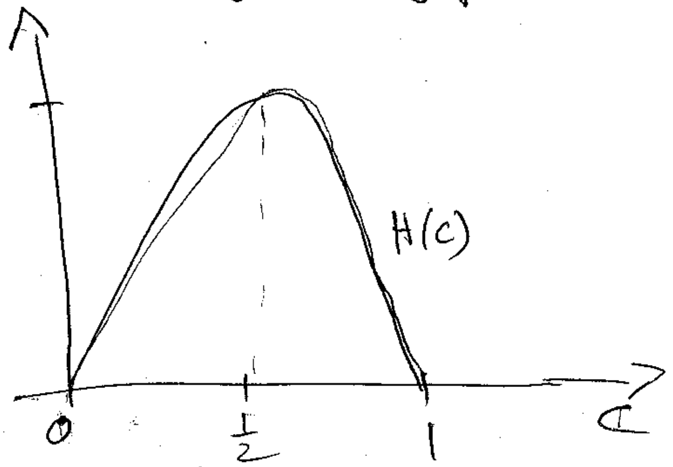


- For $k(n)=k$, $\omega(1)=k(n)=o(n)$ $\left(\frac{en}{k}\right)^k$ is a very good estimate.
- What if $k(n)=c\cdot n$ where $c>0$ is a constant?

$\rightsquigarrow \binom{n}{k}=\binom{n}{cn}=2^{H(c)n(1+o(1))}$ where $H(c)=-c\log_2 c-(1-c)\log_2(1-c)$ is the binary entropy function

How large is the largest binomial coefficient?

$\binom{n}{\frac{n}{2}}$ is the largest (n even)



The average value of binomial coefficients is

$$\frac{\sum_{i=0}^{n}\binom{n}{i}}{n+1} \Rightarrow \binom{n}{\frac{n}{2}} \geq \frac{2^n}{n+1}$$

Prop: $\frac{2^n}{\sqrt{n+1}} \geq \binom{n}{\frac{n}{2}} \geq \frac{2^n}{\sqrt{2}\sqrt{n}}$

Pf: $$\binom{n}{\frac{n}{2}} = \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} = 2^{\frac{n}{2}}\cdot 2^{\frac{n}{2}}\cdot \frac{1\cdot 3\cdot 5\cdots(n-1)\cdot \cancel{2\cdot 4\cdot 6\cdots n}}{2^{\frac{n}{2}}\left(\frac{n}{2}\right)!\;\cancel{2^{\frac{n}{2}}\left(\frac{n}{2}\right)!}} = 2^n\cdot \underbrace{\frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}}_{P}$$

$$P^2 = \frac{(1\cdot 3)\cdot(3\cdot 5)\cdot(5\cdot 7)\cdots(n-1)\cdot(n+1)}{(2\cdot 2)\cdot(4\cdot 4)\cdot(6\cdot 6)\cdots(n\cdot n)}\cdot\frac{1}{n+1} = \frac{1}{n+1}\prod_{i=1}^{\frac{n}{2}}\frac{(2i-1)(2i+1)}{(2i)^2} = \frac{1}{n+1}\prod_{i=1}^{\frac{n}{2}}\left(1-\frac{1}{(2i)^2}\right) < \frac{1}{n+1}$$

$$\| \frac{1}{2}\cdot\frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdots\frac{(n-1)\cdot(n-1)}{(n-2)\cdot n}\cdot\frac{1}{n} = \frac{1}{2n}\cdot\prod_{i=1}^{\frac{n}{2}-1}\frac{(2i+1)^2}{2i\cdot(2i+2)} = \frac{1}{2n}\prod_{i=1}\left(1+\frac{1}{2i(2i+2)}\right) > \frac{1}{2n}$$

$$\Rightarrow \frac{1}{\sqrt{n+1}} > P > \frac{1}{\sqrt{2n}} \quad \square$$

Stirling $\rightsquigarrow$ $\binom{n}{\frac{n}{2}} \approx \frac{2^n}{\sqrt{\pi}\sqrt{n}}$

Estimating the number $p(n)$ of integer partitions

$$p(n) = \left\{ (a_1, \dots, a_k) : a_1 \geq \dots \geq a_k \geq 1 \quad \sum_{i=1}^{k} a_i = n \right\}$$
$$= \sum_{i=1}^{n} p(n, i)$$

(Real Time) Exercise:

$p(5) = 7$

$5 = 1 + 1 + 1 + 1 + 1$	$1\times$
$= 2 + 1 + 1 + 1$	$4\times$
$= 2 + 2 + 1$	$3\times$
$= 3 + 1 + 1$	$3\times$
$= 3 + 2$	$2\times$
$= 4 + 1$	$2\times$
$= 5$	$1\times$
	$16 = 2^{5-1}$

How large is $p(n)$?

Easy for ordered partitions!

$$\underbrace{\bigcup_{k=1}^{n} \left\{ (a_1, \dots, a_k) \in \mathbb{N}^k : \sum_{i=1}^{k} a_i = n \right\}}_{\text{cardinality } \binom{n-1}{k-1}}$$

($k-1$ dividers to $n-1$ places between n coins)

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}$$

So $p(n) \leq 2^{n-1}$.

Better?

Generating fn of $p(n)$

Proposition: $$\sum_{n=0}^{\infty} p(n)x^n = \prod_{j=1}^{\infty} \frac{1}{1-x^j}$$

Pf: $$\prod_{j=1}^{\infty} \frac{1}{1-x^j} = \prod_{j=1}^{\infty} \left(1 + x^j + x^{2j} + x^{3j} + \cdots\right)$$

$\boxed{\text{Fix } n}$

coefficient of x^n comes from the finite product $\prod_{j=1}^{n} \frac{1}{1-x^j}$

x^{3i_3} is contribution of 3rd factor

$$= \left|\left\{(i_1, i_2, \ldots, i_n) \in \mathbb{N}_0^n : i_1 + 2i_2 + 3i_3 + \cdots + ni_n = n\right\}\right|$$

Take i_1 1s $\overbrace{1+\cdots+1}^{i_1}$

i_2 2s $\overbrace{2+\cdots+2}^{i_2}$

$\vdots$

i_n ns

$i_1 + 2i_2 + 3i_3 + \cdots = n \Rightarrow \exists \leq n$ positive terms

bijection

Then $i_1 + 2i_2 + \cdots + ni_n = a_1 + \cdots + a_n = n$

Define $i_j = |\{l : a_l = j\}|$

$$\left|\left\{(a_1, a_2, \ldots, a_n) : a_1 + \cdots + a_n = n,\ a_1 \geq \cdots \geq a_n \geq 0\right\}\right|$$

$$\| $$

$$p(n)$$

$$\prod_{j=1}^{n} \frac{1}{1-x^j} = \sum_{n=0}^{\infty} p(n)\, x^n \ge p(n)\cdot x^n \quad \boxed{\forall x \in (0,1)}$$

$$p(n) \le \frac{1}{x^n} \prod_{j=1}^{n} \frac{1}{1-x^j} \qquad \forall x \in (0,1) \rightsquigarrow \boxed{\text{Optimize!!}}$$

Which x gives the best upper bound?

Seeing a product $\rightsquigarrow$ Take logarithm!

$$\ln p(n) \le -n \ln x - \sum_{j=1}^{n} \ln(1-x^j)$$

$$-\sum_{j=1}^{n} \ln(1-x^j) = \sum_{j=1}^{n}\sum_{l=1}^{\infty} \frac{(x^j)^l}{l} = \sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=1}^{n} (x^l)^j$$

$$\le \sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=1}^{\infty} (x^l)^j = \sum_{l=1}^{\infty} \frac{1}{l} \cdot \frac{x^l}{1-x^l}$$

$$\le \sum \frac{1}{l} \cdot \frac{x^l}{(1-x)\cdot l\, x^{l-1}} = \frac{x}{1-x} \sum_{l=1}^{\infty} \frac{1}{l^2}, \qquad \sum_{l=1}^{\infty} \frac{1}{l^2} = \frac{\pi^2}{6}$$

$0 < x < 1 \Rightarrow 1+x+x^2+\dots+x^{l} \ge l\, x^{l-1}$

$$\ln p(n) \le -n \ln x + \frac{\pi^2}{6} \frac{x}{1-x}$$

Sabstitute $u = \frac{x}{1-x} \in [0,\infty) \rightsquigarrow \frac{1}{x} = 1 + \frac{1}{u}$

$$\ln p(n) \le n \ln\left(1+\frac{1}{u}\right) + \frac{\pi^2}{6} u \le n \ln e^{\frac{1}{u}} + \frac{\pi^2}{6} u = \frac{n}{u} + \frac{\pi^2}{6} \cdot u$$

Evaluate at $u = \frac{\sqrt{6n}}{\pi}$ (Optimize)

$$\ln p(n) \le \frac{n}{\frac{\sqrt{6n}}{\pi}} + \frac{\pi^2}{6} \cdot \frac{\sqrt{6n}}{\pi} = \frac{2\pi}{\sqrt{6}} \sqrt{n}$$

$$\Rightarrow \quad p(n) \le e^{\frac{2\pi}{\sqrt{6}}\sqrt{n}}$$

Exponent is only SQUARE ROOT of n

Hardy – Ramanujan

$$p(n) \approx \frac{1}{4\sqrt{3}} \cdot \frac{1}{n} \cdot e^{\frac{2\pi}{\sqrt{6}}\sqrt{n}}$$

$\frac{1}{4\sqrt{3}}$ → precise constant factor

$\frac{1}{n}$ → precise polynomial factor

$e^{\frac{2\pi}{\sqrt{6}}\sqrt{n}}$ → Dominating factor is the same as in our upper bound.

A Lower Bound showing $p(n) = e^{\Theta(\sqrt{n})}$

An unordered integer partition into k parts gives rise to atMOST k! ordered partition into k parts

$$\rightsquigarrow \quad k!\, p(n,k) \geq \binom{n-1}{k-1}$$

$$\Longrightarrow \quad p(n) \geq p(n,k) \geq \binom{n-1}{k-1} \cdot \frac{1}{k!} = \frac{(n-1)\cdots(n-k+1)}{(k-1)!\,k!}$$

$\boxed{\text{Again: } \forall k}$ Choose a k optimally!

How? Functions around extremal values often don't grow so fast.

So look for k when $\frac{(n-1)\cdots(n-k)}{k!\,(k+1)!} \approx \frac{(n-1)\cdots(n-k+1)}{(k-1)!\,k!}$

$$(n-k) \approx (k+1)k$$
$$n \approx (k+1)^2$$

CHOOSE $\boxed{k = \lfloor\sqrt{n}\rfloor}$

$$\boxed{p(n)} \geq \frac{(n-k)^{k-1}}{(k!)^2} \geq \frac{n^{k-1}\left(1-\frac{k}{n}\right)^{k-1}}{\left(\left(\frac{k}{e}\right)^k ek\right)^2} \geq \frac{n^{k-1}e^{2k-2}}{k^{2k+2}} \cdot \frac{1}{e}$$

$$= \left(\frac{n}{k^2}\right)^k \frac{e^{2k-3}}{nk^2} \geq \boxed{e^{2\sqrt{n}} \cdot \frac{1}{e^5 n^2}}$$