Bipartite graphs

A set of pairwise adjacent vertices in a graph is called a clique. A set of pairwise non-adjacent vertices in a graph is called an independent set.

A graph G is bipartite if $V(G)$ is the union of two independent sets of G. If these are disjoint, they are called the partite sets of G.

Examples. $K_{r,s}$ is bipartite, K_n is not bipartite for $n \geq 3$, P_n is bipartite for all $n \geq 1$, C_n is bipartite iff n is even (count edges leaving an independent set)

Example. The k-dimensional hypercube Q_k

$V(Q_k) = \{0,1\}^k$

$E(Q_k) = \{xy : x \text{ and } y \text{ differ in exactly one coordinate}\}$

Properties.

- $v(Q_k) = 2^k$
- Q_k is k-regular
- $e(Q_k) = k2^{k-1}$
- Q_k is bipartite

The beauty of being bipartite

Proposition. Let G be k-regular bipartite graph with partite sets A and B, $k > 0$. Then $|A| = |B|$.

Proof. Double count the edges of G by summing up degrees of vertices on each side of the bipartition.

Theorem. Every loopless multigraph G has a bipartite subgraph with at least $\frac{e(G)}{2}$ edges.

Proof by "extremality". (Consider a bipartite subgraph H with the *maximum number of edges* and prove that $d_H(v) \geq d_G(v)/2$ for every vertex $v \in V(G)$ (otherwise change H so to contradict its extremality. Finish with the Handshaking Lemma.))

Remark The constant multiplier $\frac{1}{2}$ of $e(G)$ in the Theorem is best possible.
Example: K_n. (for every bipartite $H \subseteq K_n$,

$$e(H) = i(n-i) \leq \left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor\right) = \left\lfloor \frac{n^2}{4} \right\rfloor$$

edges, which is $< (\frac{1}{2} + \epsilon)\binom{n}{2}$ for $\forall \epsilon > 0$ and large n.)

Characterization of bipartite graphs

A bipartition of G is a specification of two disjoint independent sets in G whose union is $V(G)$.

Theorem. (König, 1936) A multigraph G is bipartite iff G does not contain an odd cycle.

Proof.

$\Rightarrow$ Easy.

$\Leftarrow$ Fix a vertex $v \in V(G)$. Define sets

$$A := \{w \in V(G) : \ \exists \text{ an odd } v, w\text{-path } \}$$

$$B := \{w \in V(G) : \ \exists \text{ an even } v, w\text{-path } \}$$

Prove that A and B form a bipartition.

Lemma. Every closed odd walk contains an odd cycle.

Proof. Strong induction.

Walks, trails, paths, and cycles

A walk is an alternating list $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i.

Remark. Listing of edges is only necessary in multi-graphs.

A trail is a walk with no repeated edge.

A path is a walk with no repeated vertex.

A u, v-walk, u, v-trail, u, v-path is a walk, trail, path, respectively, with first vertex u and last vertex v.

If $u = v$ then the u, v-walk and u, v-trail is closed. A closed trail (without specifying the first vertex) is a circuit. A circuit with no repeated vertex is called a cycle.

The length of a walk trail, path or cycle is its number of edges.

Connectivity

G is connected, if there is a u, v-path for every pair $u, v \in V(G)$ of vertices.
Otherwise G is disconnected.

Vertex u is connected to vertex v in G if there is a u, v-path. The connection relation on $V(G)$ consists of the ordered pairs (u, v) such that u is connected to v.

Claim. The connection relation is an equivalence relation.

Lemma. Every u, v-walk contains a u, v-path.

The connected components of G are its maximal connected subgraphs (i.e. the equivalence classes of the connection relation).

An isolated vertex is a vertex of degree 0. It is a connected component on its own.

Cutting a graph

A cut-edge or cut-vertex of G is an edge or a vertex whose deletion increases the number of components.

If $M \subseteq E(G)$, then $G - M$ denotes the graph obtained from G by the deletion of the elements of M:

$$V(G - M) = V(G) \text{ and } E(G - M) = E(G) \setminus M.$$

For $S \subseteq V(G)$, $G - S$ obtained from G by the deletion of S and all edges incident with a vertex from S:

$$G - S := G[V(G) \setminus S].$$

For $e \in E(G)$, $G - \{e\}$ is abbreviated by $G - e$.
For $v \in V(G)$, $G - \{v\}$ is abbreviated by $G - v$.

Proposition. An edge e is a cut-edge iff it does not belong to a cycle.

Eulerian circuits

Example. How to draw the little house graph without lifting the pen?

A trail of G is called Eulerian if it contains all edges.

Proposition. In an Eulerian trail every internal vertex has even degree.
Proof. Given vertex v, pair up its incident edges.

Corollary A successful drawing of the little house graph must start at the bottom.

A multigraph is Eulerian if it has an Eulerian circuit.

Theorem. Let G be a connected multigraph. Then

G is **Eulerian** iff $d(v)$ is **even** for $\forall v \in V$

Proof.

$\Rightarrow$ Follows from Proposition.

$\Leftarrow$ Extremality: Consider longest trail T in G and prove that: (i) T is closed, (ii) $V(T) = V(G)$, (iii) $E(T) = E(G)$.

Beginnings of Graph Theory

1735: Euler and the Königsberg's bridges