ELEMENTARY COUNTING PRINCIPLES

- RULE OF SUM: $S = \dot{\bigcup}_{i=1}^{n} S_i \implies |S| = \sum_{i=1}^{n} |S_i|$

  ↓ S_i are pairwise disjoint

  i.e., $\forall i \neq j \; S_i \cap S_j = \emptyset$

- Basis of every CASE ANALYSIS [Classify elements according to some property. Cases should • be disjoint, • cover everything]

Example: 1st grader "word problem"

Drawer with 8 pairs of yellow socks
5 —"— blue socks
3 —"— green socks
and no more.
How many socks are in the drawer?

Example: # of k-element subsets of an n-element set

$k, n \in \mathbb{N}_0$

Notation: $[n] := \{1, 2, \dots, n\}$ $\quad n \in \mathbb{N}$

X set, $k \in \mathbb{N}_0$: $\binom{X}{k} := \{K \subseteq X : |K| = k\}$

$\binom{0}{0} := 1 =$ # of subsets of $\emptyset$

$\binom{n}{k} := \left|\binom{[n]}{k}\right| =$ # of k-element subsets of $[n]$

Example: $\binom{[4]}{3} = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$

Proposition (Pascal recurrence)

$\forall \; n \geq k \geq 1$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Remark: We do not "know" a formula for $\binom{n}{k}$ yet ...

Pf: Classify ~~#~~ k-subsets of $[n]$ according to whether they contain n or not.

$$S_1 = \left\{ T \in \binom{[n]}{k} : n \in T \right\}$$

$$S_2 = \left\{ T \in \binom{[n]}{k} : n \notin T \right\}$$

- $S_1 \cap S_2 = \emptyset$
- $S_1 \cup S_2 = \binom{[n]}{k}$

Sum Rule $\Rightarrow$ $\binom{n}{k} = \left|\binom{[n]}{k}\right| = |S_1| + |S_2|$

$$|S_2| = \left|\binom{[n-1]}{k}\right| = \binom{n-1}{k}$$

Bijection

$$S_1 \longleftrightarrow \binom{[n-1]}{k-1} \Rightarrow |S_1| = \left|\binom{[n-1]}{k-1}\right| = \binom{n-1}{k-1}$$

$$T \in S_1 \longleftrightarrow T \setminus \{n\} \in \binom{[n-1]}{k-1}$$

$\square$

Pascal triangle: Values can be calculated using recurrence and $\binom{n}{0} = \binom{n}{n} = 1 \ \forall n \in \mathbb{N}_0$

```
        1
       1 1
      1 2 1
     1 3 3 1
    1 4 6 4 1
   1    6+4=10    1
  1                 1
```

Rule of Product

$$S = \bigtimes_{i=1}^{n} S_i \implies |S| = \prod_{i=1}^{n} |S_i|$$

$$\{(a_1, \dots, a_n) : a_i \in S_i \ \forall i = 1, 2, \dots, n\}$$

(Pf: SumRule ⇒ Classify according to first coordinate)

Example: # of bitstrings of length n is 2^n

"0/1 words"

$$\left|\{(a_1, \dots, a_n) : a_i \in \{0,1\}\}\right| = \prod_{i=1}^{n} |\{0,1\}| = 2^n$$

Example: What is $\binom{n}{k}$?

To start: Def: set X, $k \in \mathbb{N}$ • k-permutation of X is an injective $f: [k] \to X$

(Alternative ways to write the same thing:
- vector of length k with pairwise distinct entries from X $(f(1), f(2), \dots, f(k))$
- short: $f(1)f(2)\dots f(k)$ write as word)

- n-permutation is an n-permutation of $[n]$
- permutation of X is an $|X|$-permutation of X

Notation $X^{\underline{k}} = \{(a_1, \dots, a_k) \in X^k : a_i \neq a_j \ \forall i \neq j\}$

$n^{\underline{k}} := n(n-1)\cdot\ldots\cdot(n-k+1) = \prod_{i=0}^{k-1}(n-i)$, $n^{\underline{0}} = 1$

Proposition $\forall n, k \in \mathbb{N}_0$

$$\left|[n]^{\underline{k}}\right| = n^{\underline{k}}$$

Pf: k-permutation: Let $1 \le k \le n$

$$(\circ, \circ, \circ, \ldots\ldots, \circ) \in [n]^k$$

- first entry: can be chosen n ways
- second entry: can be chosen $n-1$ ways (INDEPENDENT OF what was chosen as first coordinate)
- third entry: can be chosen $n-2$ ways (INDEPENDENT of what was chosen as first AND second entry)
- last entry: can be chosen $n-k+1$ ways (INDEPENDENT of what was chosen before)

$$n(n-1)\cdot(n-2)\cdot \_\_\_ \cdot (n-k+1) = \prod_{i=0}^{k-1}(n-i) = n^{\underline{k}}$$

$$n^{\underline{0}} = 1 \qquad \text{(empty permutation)}$$

$\square$

- GENERALIZED PRODUCT RULE

We are not counting a product anymore.

Question i (What is the i^{th} entry?) has the same # of possible answers for every sequence of answers to the first $i-1$ questions.

Canonical map $X^k \xrightarrow{F} \binom{X}{k}$ $\qquad X^k = \bigcup\limits_{T \in \binom{[n]}{k}} F^{-1}(T)$

$$(a_1, \ldots, a_k) \longrightarrow \{a_1, \ldots, a_k\}$$

$F^{-1}(T)$ = permutations of T

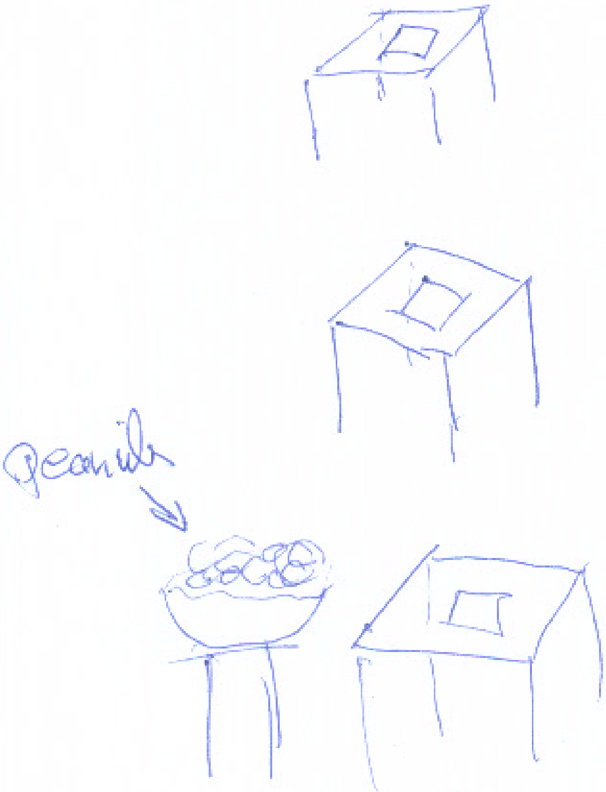
$$n^{\underline{k}} = \left|[n]^{\underline{k}}\right| = \sum_{T \in \binom{[n]}{k}} \left|F^{-1}(T)\right| \quad (\left|F^{-1}(T)\right| = k!)$$

$$= \left|\binom{[n]}{k}\right| \cdot k!$$

$$\Rightarrow \boxed{\binom{n}{k} = \frac{n^{\underline{k}}}{k!}}$$

Example: 6 people

How many ways are there to seat them to play chess at 3 boards?



FORMULATE A PRECISE QUESTION!!!

- Does it matter who plays black or white?

- Does it matter who sits next to the peanuts?

For us: NO and NO

So we enumerate the set $\{\{A_1, A_2, A_3\} : A_1 \cup A_2 \cup A_3 = [6],\ |A_1| = |A_2| = |A_3| = 2\}$

- Choose A_1 $\binom{6}{2}$ ways
- Once A_1 is chosen $\exists\ \left|\binom{[6] \setminus A_1}{2}\right| = \binom{4}{2}$ ways to choose A_2
- A_1 and A_2 are chosen $\Rightarrow$ $A_3 = [6] \setminus (A_1 \cup A_2)$ unique

Every set $\{A_1, A_2, A_3\}$ was counted $3! = 6$ ways this way.

So: $\dfrac{\binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2}}{3!} = \dfrac{15 \cdot 6 \cdot 1}{6} = \underline{\underline{15}}$

Another way: Alice, Bob, Carole, Daniel, Emma, Frank

Question #1: Who is the partner of Alice?

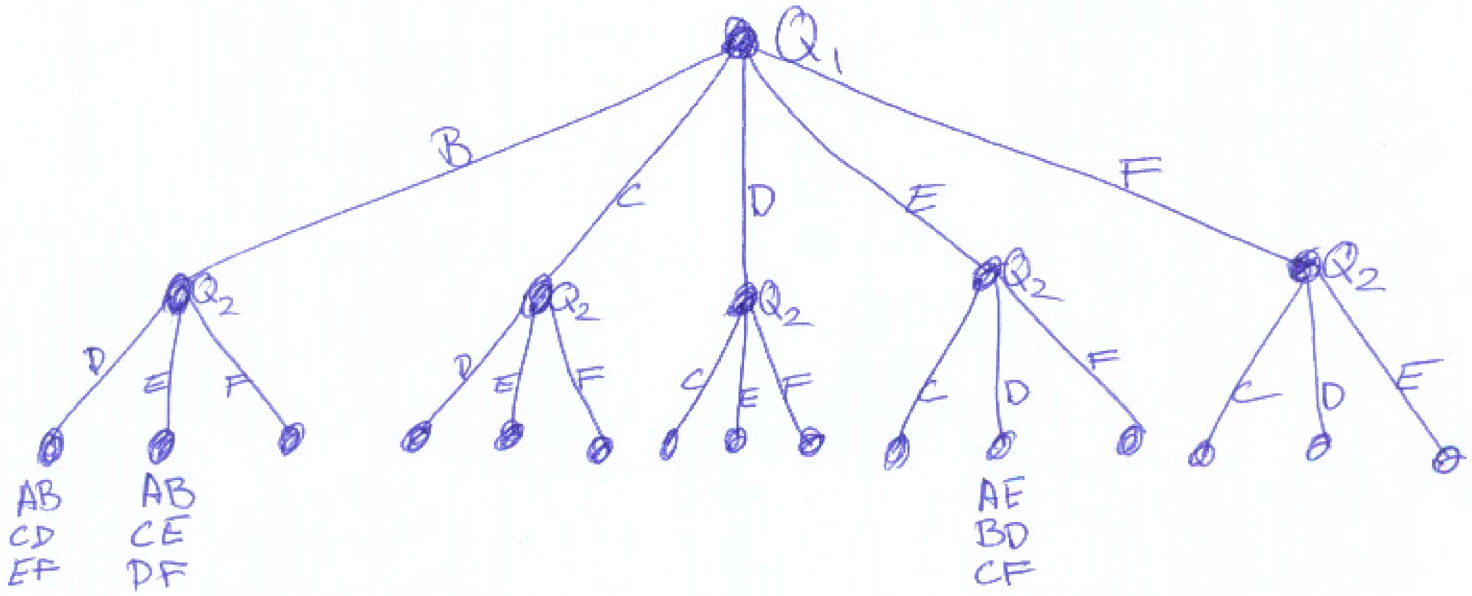
5 possible answers

Question #2: Who is the partner of the person whose name starts with the letter coming first in the alphabet, ~~among these~~ after removing Alice and her partner from consideration?

3 possible answers — independent of the first answer

(**Question #3:** ~~Who is the~~ ....

1 possible answer unique pair remains)



So $5 \cdot 3 = 15$ ways, different answers lead to different pairings

(Same answer as for first solution. phew....)

In general, to distribute $n=2k$ people into k pairs

Two solutions $\rightsquigarrow$ identity

$$\frac{\binom{n}{2}\cdot\binom{n-2}{2}\cdot\ldots\cdot\binom{2}{2}}{\left(\frac{n}{2}\right)!} = (n-1)(n-3)\cdot\ldots\cdot(n-(2i+1))\cdot\ldots\cdot 3\cdot 1 = (n-1)!!$$

Question #i: Who is the partner of the person whose name comes first in the alphabet among those who are not paired yet?

Key: While the SET of possible answers MIGHT depend on the answers to the first $i-1$ questions, the NUMBER of possible answers does NOT!

Rule of Bijection: If $\exists$ bijection $F: S \to T \Rightarrow |S| = |T|$

Example ① # of subsets of $[n]$?

$$2^{[n]} := \{T \subseteq [n]\}$$

bijection

$$T \xrightarrow{F} v_T \in \{0,1\}^n \qquad (v_T)_i = \begin{cases} 0 & \text{if } i \notin T \\ 1 & \text{if } i \in T \end{cases}$$

$T \in 2^{[n]}$

- $v_T \in \{0,1\}^n$ indeed
- F is injective: $T \neq T' \Rightarrow \exists i \in (T \setminus T') \cup (T' \setminus T)$ $\Rightarrow (v_T)_i \neq (v_{T'})_i$
- F is surjective: For $\forall v \in \{0,1\}^n$ $\exists$ a set $T \subseteq [n]$ s.t. $v_T = v$
  $T = \{i \in [n] : v_i = 1\}$

So $$|2^{[n]}| \underset{\text{Bijection}}{=} |\{0,1\}^n| \underset{\text{Product Rule}}{=} |\{0,1\}|^n = 2^n$$

Example ② We had bijection already

$$S_1 = \{T : T \in \binom{[n]}{k}, \ n \in T\} \longrightarrow \binom{[n-1]}{k-1}$$

$$T \longrightarrow T \setminus \{n\}$$

Example ③ What is $\binom{n}{k}$?

Bijection

$$[n]^k \xrightarrow{\quad F \quad} \binom{[n]}{k} \times [k]^k$$

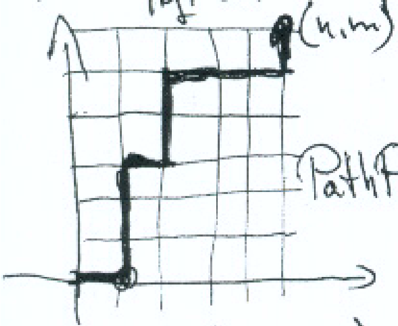
$$(a_1, \dots, a_k) \longrightarrow \left(\{a_1, \dots, a_k\},\ \pi\right)$$

π is such that
$\pi : [k] \longrightarrow [k]$
$a_{\pi(1)} < a_{\pi(2)} < \dots < a_{\pi(k)}$

$$\Rightarrow \quad n^k = \left|[n]^k\right| = \left|\binom{[n]}{k} \times [k]^k\right| = \left|\binom{[n]}{k}\right| \cdot \left|[k]^k\right| = \binom{n}{k} \cdot k!$$

$$\Rightarrow \quad \binom{n}{k} = \frac{n^k}{k!}$$

Example ④ # Lattice paths from $(0,0)$ to (n,m) $=: L(n,m) = \binom{n+m}{n}$



Encoding

Path P $\longrightarrow$ $\overbrace{\text{RUUURUURRRU}}^{w}$ $\longrightarrow$ ~~$\binom{[n+m]}{n}$~~ $\{i : w_i = R\}$

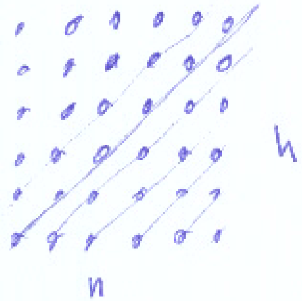
$$L(n,m) \xrightarrow{\text{Bijection}} \left\{\begin{array}{l}\text{R/U words with} \\ n \text{ Rs and} \\ m \text{ Us}\end{array}\right\} \longrightarrow \binom{[n+m]}{n}$$

- $F(P) \in \binom{[n+m]}{n}$ since $\forall$ Path has exactly n Right move, exactly m Up move
- F is injection ($P \neq P' \Rightarrow F(P)$ differs from $F(P')$ in the element corresponding to the first time P separates from P')
- F is surjection ($\forall$ n Right move and m Up move in ANY order takes a path from $(0,0)$ to (n,m))

Double Counting (Rule of counting two ways)

- When two formulas enumerate the same set they must be equal.
- Exchange of summation (Finite Fubini)

Count gridpoints in



(1) vertical line by vertical line $n \cdot n = n^2$

(2) diagonal by diagonal

$1+2+3+4+\dots+n-1+n+n-1+n-2+\dots+2+1$

$= 2(1+2+\dots+n-1)+n$

$$\Rightarrow \quad n^2 = 2\sum_{i=1}^{n-1} i + n$$

$$\Rightarrow \binom{n}{2} = \frac{n^2-n}{2} = \sum_{i=1}^{n-1} i$$

Example: Number theoretic fn $d : \mathbb{N} \to \mathbb{N}$

$d(n) = \#$ of divisors of n

Real time exercise: Evaluate $d(n)$ for $1 \le n \le 8$

$d(1) = 1$

$d(2) = 2$

$d(3) = 2$

$d(4) = 3$

$d(5) = 2$

$d(6) = 4$

$d(7) = 2$

$d(8) = 4$

$d(8191) = 2$

$d(8192) = 14$

d jumps up and down

What is the average value?

$$\bar{d}(n) = \frac{\sum_{i=1}^{n} d(i)}{n}$$

$\bar{d}(1) = 1$

$\bar{d}(2) = \frac{3}{2}$

$\bar{d}(3) = \frac{5}{3}$

$\bar{d}(4) = 2$

$\bar{d}(5) = 2$

$\bar{d}(6) = \frac{7}{3}$

$\bar{d}(7) = \frac{16}{7}$

$\bar{d}(8) = \frac{5}{2}$

Double counting to the rescue

$$\sum_{i=1}^{n} d(i) = \sum_{i=1}^{n} \sum_{\substack{1\le j\le n \\ j \mid i}} 1 = \left| \left\{ (j,i) : \begin{matrix} i\in[n], j\in[n] \\ j \mid i \end{matrix} \right\} \right|$$

$$= \sum_{j=1}^{n} \sum_{\substack{1\le i\le n \\ j\mid i}} 1 = \sum_{j=1}^{n} \left\lfloor \frac{n}{j} \right\rfloor$$

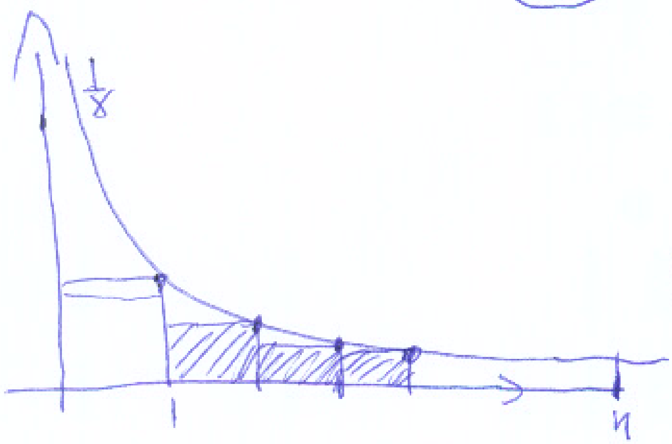
Exchange summation

In this sum j is fixed and we count the # multiples of j up to n

Estimate

$$n(H_n - 1) = \sum_{j=1}^{n} \left(\frac{n}{j} - 1\right) \le \sum_{j=1}^{n} \left\lfloor \frac{n}{j} \right\rfloor \le \sum_{j=1}^{n} \frac{n}{j} = n \underbrace{\sum_{j=1}^{n} \frac{1}{j}}_{H_n}$$

H_n Harmonic number



$$H_{n-1} = \sum_{j=1}^{n-1} \frac{1}{j} \ge \underbrace{\int_1^n \frac{1}{x}dx}_{\ln n} \ge \sum_{j=2}^{n} \frac{1}{j} = H_n - 1$$

So

$$H_n - 1 \le \frac{n(H_n-1)}{n} \le \underbrace{\frac{\sum_{i=1}^{n} d(i)}{n}}_{\overline{d}(n)} \le \frac{nH_n}{n} = H_n$$

Binomial Identities

- $\binom{n}{k} = \binom{n}{n-k}$ Pf: Bijection $A \xrightarrow{F} \overline{A}$, with $A \in \binom{[n]}{k}$ and $\overline{A} \in \binom{[n]}{n-k}$

Sum of a row in Pascal's Δ

- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ Pf: Sum Rule: Classify subsets of $[n]$ according to size

$$\bigcup_{k=0}^{n} \binom{[n]}{k} = 2^{[n]}$$

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \left|\binom{[n]}{k}\right| = \left|\bigcup \binom{[n]}{k}\right| = \left|2^{[n]}\right| = 2^n$$

Binomial Thm:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Pf: Induction on n (and recurrence)

Applications: $x = y = 1 \Rightarrow 2^n = (1+1)^n = \sum \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$

$x = 1,\ y = -1$ $0^n = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k (-1)^{n-k} = \sum_{j=0}^{n} (-1)^j \binom{n}{j}$

$$\Rightarrow \sum_{\substack{0 \le j \le n \\ \text{even}}} \binom{n}{j} = \sum_{\substack{0 \le j \le n \\ j \text{ odd}}} \binom{n}{j}$$

# of subsets of even size = # of subsets of odd size

UNLESS $n = 0$ ($0^0 = 1$, $\emptyset$ has 1 subset)

Alternative combinatorial proof:

$$\mathcal{O} = \{T \subseteq [n] : |T| \text{ odd}\}$$

$$\mathcal{E} = \{T \subseteq [n] : |T| \text{ even}\}$$

$$A \xrightarrow{F} \overline{A}$$

$$A \in \mathcal{O}, \quad \overline{A} \in \mathcal{E}$$

works if n is odd

(F is bijection)

In general:

$$A \longrightarrow \begin{cases} A \setminus \{n\} & \text{if } n \in A \\ A \cup \{n\} & \text{if } n \notin A \end{cases}$$

$$A \in \mathcal{O}, \quad \text{image} \in \mathcal{E}$$

bijection $\Longrightarrow |\mathcal{O}| = |\mathcal{E}|$

Further identities (HW?)

• $\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}$ (Classify, according to largest element, the (k+1)-subsets of [n+1])

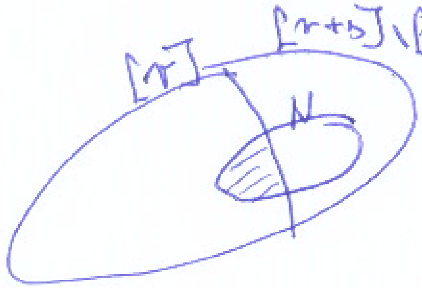
of (k+1)-subsets of [n+1] with largest element i+1.

• $\sum_{i=0}^{n} \binom{m+i}{i} = \binom{m+n+1}{n}$ (HW?)

• Vandermonde Identity $\forall r,s$

$$\sum_{k=0}^{k} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$$

Classify n-subsets of [r+s] according to size of intersection with [r]



Map

$$N \longrightarrow (N\cap[r],\ N\cap([r+s]\setminus[r]))$$

Bijection between

$$R\cup S \longleftarrow (R,\ S)$$

$$\binom{[r+s]}{n} \longrightarrow \bigcup_{k=0}^{n} \binom{[r]}{k} \times \binom{[r+s]\setminus[r]}{n-k}$$

• $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$ (HW?)