# Setup for the next days.

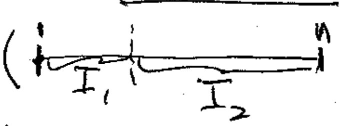
| | |
|---|---|
| sequence $(a_0, a_1, a_2, \dots)$ | $a_n :=$ # ways to build a structure of Type I on $[n]$ |
| sequence $(b_0, b_1, b_2, \dots)$ | $b_n :=$ # ways to build a structure of Type II on $[n]$ |

Examples for "building structure on $[n]$"
- choosing an ordering $(n!)$
- choosing a subset $(2^n)$
- choosing an element $(n)$
- ⋮
- doing nothing $(1)$
- $\emptyset$ $(0)$

What do the coefficients of the series $C(x) = A(x)B(x)$ represent? (where $A(x) := \sum_{n=0}^{\infty} a_n x^n$ $B(x) = \sum_{n=0}^{\infty} b_n x^n$)

Proposition (Product Formula) $C(x) = A(x) \cdot B(x) := \sum c_n x^n$

$\Rightarrow$ $C_n =$ # of ways to do the following: (maybe empty)

① Split $[n]$ into two intervals $I_1$ and $I_2$ ()



② Build Structure of Type I on $I_1$

③ —"— Type II on $I_2$

Pf: Classify according to where do we split: $[i] \cup ([n] \setminus [i])$

$c_n^{(i)} :=$ # possibilities if we split at $i$ $= a_i \cdot b_{n-i}$ (Product Rule)

# of ways $= \sum_{i=0}^{n} c_n^{(i)}$ (Sum Rule) $\Rightarrow$ # of ways $= \sum_{i=0}^{n} a_i b_{n-i}$

Also: $(a_0 + a_1 x + a_2 x^2 + \dots)(b_0 + b_1 x + b_2 x^2 + \dots) \Rightarrow \sum (\underbrace{a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0}_{= c_n}) x^n$

Example: Design term in an Engineering Department

Term: $n$ days

Decisions: • How long should theoretical part be?

$k$ days $(1 \le k \le n-2)$

Rest is laboratory part ($n-k$ days)

• When should be the

× project day for the theoretical part?

× two project days for the lab part?

How many ways are there? $\boxed{f_n}$

Product Formula:

$a_n :=$ # ways to select project day from a $n$ day long theoretical part

$a_n = n$

$b_n :=$ # ways to select TWO project days from an $n$ day long lab part

$b_n = \binom{n}{2}$

$$a(x) = \sum_{n=1}^{\infty} n x^n = x \cdot \left(\sum_{n=0} x^n\right)' = x \cdot \left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2}$$

$$b(x) = \sum_{n=2}^{\infty} \binom{n}{2} x^n = \frac{x^2}{2} \cdot \left(\sum_{n=0}^{\infty} x^n\right)'' = x^2\left(\frac{1}{1-x}\right)'' = \frac{x^2}{(1-x)^3}$$

$$f(x) = \sum f_n x^n = a(x) \cdot b(x) = \frac{x^3}{(1-x)^5} = x^3 \sum_{n=0}^{\infty} \binom{n+4}{4} x^n = \sum_{n=3}^{\infty} \binom{n+1}{4} x^n$$

$$\Longrightarrow \boxed{f_n = \binom{n+1}{4}}$$

# Catalan numbers

Student   empty jar

every day : either 1 Euro coin in
or 1 Euro coin out

After $2n$ days the jar is empty again.
How many ways can this happen? $c_n$

Define $c_0 := 1$ $\boxed{\text{Real time exercise } c_1, c_2, c_3}$

$$c_n := \left| \left\{ (b_1, b_2, \dots, b_{2n}) \in (+1,-1)^{2n} : \sum_{j=1}^{2n} b_j = 0 \text{ AND } \forall k \sum_{j=1}^{k} b_j \geq 0 \right\} \right|$$

At the end the jar is empty

The jar never has negative coins amount

Classify ~~all the~~ elements according to ~~FIRST~~ SMALLEST $k>0$ when $\sum_{j=1}^{2k} b_j = 0$ (FIRST time when jar is empty again)

# of ways to finish afterwards : $c_{n-k}$

# of ways to get there : MUST start with $b_1 = +1$
finish with $b_{2k} = -1$

AND $\boxed{\sum_{j=2}^{l} b_j \geq 0 \quad \forall \; l, \; 2 \leq l \leq 2k}$ $\boxed{\text{AND } \sum_{j=2}^{2k-1} b_j = 0}$

There are $c_{i-1}$ of these.

Recurrence: $c_n = \sum_{i=1}^{n} c_{i-1} c_{n-i} = \sum_{j=0}^{n-1} c_j c_{n-1-j}$

$$C(x)\cdot C(x) = \sum_{n=0}^{\infty} \underbrace{\sum_{i=0}^{n} c_i c_{n-i}}_{c_{n+1}} x^n = \frac{C(x)-c_0}{x}$$

$$x C(x)^2 - C(x) + 1 = 0$$

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Is it $C(x) = \dfrac{1+\sqrt{1-4x}}{2x}$ or $\boxed{C(x) = \dfrac{1-\sqrt{1-4x}}{2x}}$

CANNOT BE BOTH

when $x \to 0$ $+\infty$ But it should tend to $\underline{\underline{c_0 = 1}}$ So this is the real thing

$$\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n = 1 + \frac{1}{2}\cdot(-4x) + \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{2!}(-4)^2 x^2 + \dots$$

$$\dots + \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\dots\left(\frac{1}{2}-n+1\right)}{n!}(-4)^n x^n + \dots$$

$$= 1 - 2x - 2x^2 \dots (-1)^{2n-1} \frac{(2n-3)!!}{\cancel{2^n}\, n!} \cancel{4^n}^{\,2^n} x^n$$

$$\frac{1-\sqrt{1-4x}}{2x} = 1 \cancel{-2x} \dots + \frac{(2n-3)!!}{n!} \frac{2^{n-1}(n-1)!}{(n-1)!} x^{n-1}$$

$2^{n-1}(n-1)! = (2n-2)!!$

$$\boxed{c_n} = \frac{1}{n}\frac{(2n-2)!}{(n-1)!(n-1)!} = \boxed{\frac{1}{n}\binom{2n-2}{n-1}}$$

# Putting Catalan numbers into the Product Formula setup

<u>Define:</u>
- good sequence $(b_1,\dots,b_{2n}) \in \{-1,1\}^{2n}$: $\sum_{i=1}^{2n} b_i = 0$
  $\forall j,\ 1 \le j \le 2n \quad \sum_{i=1}^{j} b_i \ge 0$
- very good sequence $(v_1,\dots,v_{2n}) \in \{-1,1\}^{2n}$: $\sum_{i=1}^{2n} v_i = 0$
  $\forall j,\ 2 \le j \le 2n-1 \quad \sum_{i=1}^{j} b_i > 0$

$\boxed{\forall n \ge 1}$ $c_n :=$ # of <u>good</u> sequences of length $2n$

$v_n :=$ # of <u>very good</u> — " —

We have shown $c_n = \sum_{i=1}^{n} v_i c_{n-i}$ (provided we define $\boxed{c_0 := 1}$

For Product Formula we need $\boxed{c_n = \sum_{i=0}^{n} v_i c_{n-i}}$ so define $\boxed{v_0 := 0}$

$\boxed{n \ge 1}$

- Split $[n]$ into intervals $I_1, I_2$
- Put very good sequence on $I_1$ (Structure of Type I)
- Put good sequence on $I_2$ ( — " — Type II)

$\Rightarrow$ Generating function is $C(x) \cdot V(x)$

It is equal to $C(x)$ <u>EXCEPT</u> $n=0$ ($c_0 = 1$, but $c_0 \cdot v_0 = 0$)

So $C(x) \cdot V(x) = C(x) - 1$

Now $V(x) = x\,C(x)$, since we have shown $\boxed{v_n = c_{n-1} \ \forall n \ge 1}$

Indeed, there is a bijection between good sequences of length $2n-2$ and very good sequences of length $2n$

$(v_1, \dots, v_{2n})$ very good $\longrightarrow$ $(v_2, \dots, v_{2n-1})$ is good

$(+1, b_1, \dots, b_{2n-2}, -1)$ very good $\longleftarrow$ $(b_1, \dots, b_{2n-2})$ good

<u>Prop</u> (k-wise Product Formula) For $j = 1, 2, \dots, k$, let $(a_0^{(j)}, a_1^{(j)}, a_2^{(j)}, \dots)$ sequence will

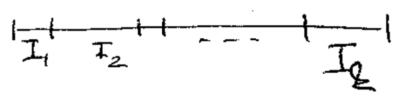
$a_n^{(j)} := \#$ of ways to build Structure of Type $j$ on $[n]$

let $A_j(x) = \sum_{n=0}^{\infty} a_n^{(j)} x^n$ and let $(c_0, c_1, c_2, \dots)$ be the coefficients in $C(x) = \sum c_n x^n = \prod_{j=1}^{k} A_j(x)$

<u>Then</u> $c_n = \#$ of ways to do the following:

(1) Split $[n]$ into $k$ intervals $I_1 \cup I_2 \cup \dots \cup I_k$

$\boxed{\text{maybe empty}}$




(2) Build structure of Type $j$ on $I_j$

Pf: Induction on $k$

~~Then~~ $k = 1$ ✓ ($k = 2$ ✓ deg Product Form)

Do the procedure as follows:

(1) Split $[n]$ into <u>two</u> intervals $I^*$ and $I_k$

(2) Build the "(k-1) interval"-structure with Structures of Type $1, 2, \dots, k-1$ on $I^*$

(3) Build Structure of Type $k$ on $I_k$

Generating function of the sequence "How many ways can we do (2) if $|I^*| = n$?" is $\boxed{\prod_{j=1}^{k-1} A_j(x)}$

Generating function of (3) is $A_k(x) \Longrightarrow$ By Product Formula $\rightsquigarrow \prod_{j=1}^{k} A_j(x)$

# Composition of Generating Functions

Recall setup: $(a_n)$, $(b_n)$, Structure of Type I, Type II, $A(x)$, $B(x)$

$\boxed{\text{What do the coefficients of } B(A(x)) \text{ represent?}}$

For $B(A(x))$ to make sense, we require $\boxed{a_0 = 0}$

Look at first Special Case when $(b_0, b_1, \dots) = (1, 1, 1, \dots)$

that is when $B(x) = \frac{1}{1-x}$

**Prop:** $\boxed{\frac{1}{1-A(x)}}$ is the generating fn of the sequence

$\boxed{\forall n \geq 1}$ $h_n := $ # of ways to do the following

① split $[n]$ into nonempty intervals (unspecified number)

② build Structure of Type I on each

Define $\boxed{h_0 := 1}$

**Pf:** Classify according to how many nonempty intervals are in the partition: $k$

$h_n^{(k)} :=$ # of choices with $k$ non-empty intervals in partition

generating function of $h_n^{(k)}$: $\sum_{n=0}^{\infty} h_n^{(k)} x^n = (A(x))^k$ ← k-wise Product Form

Define $\boxed{h_0^{(k)} := 0}$

This is true because $a_0 = 0$, so partitions with empty intervals are not counted. (In the ~~sequence~~ k-wise Product Theorem partitions ~~also~~ with empty intervals were allowed.)

Sum Rule $\Rightarrow h_n = \sum_{k=1}^{n} h_n^{(k)} = \sum_{k=1}^{\infty} h_n^{(k)}$ since $h_n^{(k)} = 0$ if $k > n$

$$\Rightarrow H(x) = \sum_{n=0}^{\infty} h_n x^n = h_0 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} h_n^{(k)} x^n = 1 + \sum_{k=1}^{\infty} \underbrace{\sum_{n=1}^{\infty} h_n^{(k)} x^n}_{(A(x))^k}$$

__Example:__ $n$ soldiers of a squadron stand in line

Sergeant:
- splits the line at some places to form smaller units
- names a commander for each unit

$h_n :=$ # of ways this is possible. Define $\underline{h_0 := 1}$

Use Proposition: $\boxed{a_k := k}$ $\forall k \geq 0$ (# of ways to select commander from $[k]$)

( Note $\underline{\underline{a_0 := 0}}$ )

$$A(x) = \sum_{k=0}^{\infty} k x^k = x \cdot \sum_{k=0}^{\infty} k x^{k-1} = x \left( \sum_{k=0}^{\infty} x^k \right)' = x \left( \frac{1}{1-x} \right)' = \frac{x}{(1-x)^2}$$

$$\Rightarrow \sum_{n=0}^{\infty} h_n x^n = H(x) = \frac{1}{1-A(x)} = \frac{1}{1 - \frac{x}{(1-x)^2}} = \frac{(1-x)^2}{1-3x+x^2} = 1 + \frac{x}{1-3x+x^2}$$

$$\frac{x}{1-3x+x^2} = \frac{A}{1-\alpha_1 x} + \frac{B}{1-\alpha_2 x}$$

$$\alpha_1 = \frac{\sqrt{5}+3}{2} \qquad \alpha_2 = \frac{-\sqrt{5}+3}{2}$$

$$z^2 - 3z + 1 = 0$$

$$\alpha_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

$A + B = 0$ and $-\alpha_2 A - \alpha_1 B = 1$

$$-\alpha_2 A + \alpha_1 A = 1$$

$$B = -A = \frac{1}{\sqrt{5}} \Leftarrow A = \frac{1}{\alpha_1 - \alpha_2} = \frac{1}{\sqrt{5}}$$

$$\Rightarrow H(x) = 1 + \frac{1}{\sqrt{5}} \left( \frac{1}{1-\alpha_1 x} - \frac{1}{1-\alpha_2 x} \right) = 1 + \frac{1}{\sqrt{5}} \left( \sum_{n=0}^{\infty} (\alpha_1 x)^n - \sum_{n=0}^{\infty} (\alpha_2 x)^n \right)$$

$$= 1 + \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \alpha_1^n x^n - \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \alpha_2^n x^n$$

$$\Longrightarrow h_n = \frac{1}{\sqrt{5}} \left( \left( \frac{3+\sqrt{5}}{2} \right)^n - \left( \frac{3-\sqrt{5}}{2} \right)^n \right) \text{ and } h_0 = 1$$

Prop (General Composition Formula)

Recall setup: $(a_n)$, $(b_n)$, Structures of Type I, II, $A(x)$, $B(x)$ $\boxed{a_0=0}$

$B(A(x))$ is the generating function of the sequence

$g_n :=$ # of ways to do the following

1. split $[n]$ into non-empty intervals (unspecified number)
2. build Structure of Type I on each
3. build Structure of Type II on the interval of intervals

Pf: $g_0 = b_0$ ✓ For $n \geq 1$

Again: Classify according to number of nonempty intervals in partition

$h_n^{(k)}$: # of ways to do steps 1 and 2 with $k$ non-empty intervals

- Then $g_n = \sum_{k=0}^{\infty} h_n^{(k)} \cdot b_k$ Define $\boxed{h_0^{(k)} := 0}$

# of ways to do Step 3, no matter how Step 1 and 2 happened with $k$ intervals ("Product Rule")

- $\forall k \geq 1$: $\sum_{n=0}^{\infty} h_n^{(k)} x^n = (A(x))^k$

Product Formula (again, using $a_0=0$)

$$\Rightarrow \sum_{n=0}^{\infty} g_n x^n = g_0 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} h_n^{(k)} \cdot b_k x^n = g_0 + \sum_{k=1}^{\infty} b_k \sum_{n=0}^{\infty} h_n^{(k)} \cdot x^n$$

$$\sum_{n=0}^{\infty} h_n^{(k)} \cdot x^n = (A(x))^k$$

$$= b_0 + \sum_{k=1}^{\infty} b_k (A(x))^k = B(A(x))$$

__Example__ for full Composition Formula

$n$ soldiers in line

- line split at some places forming smaller units
- a subset (possibly empty) of the units is chosen for night duty

$a_n = 1 \quad \forall n$ (No structure on individual units)

$a_0 = 0$

$b_n = 2^n$ (# of subsets of $[n]$)

$$A(x) = \sum_{k=1}^{\infty} x^k = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

$$B(x) = \sum_{k=0}^{\infty} 2^k x^k = \sum_{k=0}^{\infty} (2x)^k = \frac{1}{1-2x}$$

$$B(A(x)) = \frac{1}{1 - \frac{2x}{1-x}} = \frac{1-x}{1-3x} = \frac{1}{1-3x} - \frac{x}{1-3x}$$

$$= \sum_{i=0}^{\infty} (3x)^i - x \sum_{i=0}^{\infty} (3x)^i = 1 + \sum_{i=1}^{\infty} \underbrace{\left(3^i - 3^{i-1}\right)}_{2 \cdot 3^{i-1}} x^i$$

For $n \geq 1$ soldiers $\Rightarrow \exists\ 2 \cdot 3^{n-1}$ options

# Exponential generating fns

When the sequence $(a_0, a_1, a_2, \ldots)$ grows too fast...
(superexponentially: $a_n \gg K^n \ \forall K \in \mathbb{R}$)

**Def:** $(a_0, a_1, \ldots)$ sequence of reals

$$\hat{A}(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$ is the exponential generating function of the sequence

**Example:** $a_0 = 1$, $\forall n \geq 0 \quad a_{n+1} = (n+1) a_n - n^2 + 1$
(linear recurrence, but coefficient is NOT constant!)

Closed formula?

$$\frac{n-1}{n!} = \begin{cases} [illegible] & n=0 \\ \frac{1}{(n-1)!} - \frac{1}{n!} & n \geq 1 \end{cases}$$

$$\sum_{n=0}^{\infty} a_{n+1} \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} (n+1) \frac{a_n}{(n+1)!} x^{n+1} - \sum_{n=0}^{\infty} \frac{n^2-1}{(n+1)!} x^{n+1}$$

$$= x \cdot \sum a_n \frac{x^n}{n!} - \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n-1)!} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$$= x \cdot \hat{A}(x) - x^2 e^x + x e^x$$

$$\hat{A}(x) - a_0 \frac{x^0}{0!}, \qquad a_0 \frac{x^0}{0!} = 1$$

$$\hat{A}(x)(1-x) = 1 - x^2 e^x + x e^x$$

$$\hat{A}(x) = \frac{1}{1-x} + x e^x = \sum_{n=0}^{\infty} x^n + \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$

$$\Rightarrow \frac{a_n}{n!} = 1 + \frac{1}{(n-1)!} \quad \Rightarrow a_n = n! + n$$

$\forall n \geq 1$

# Products of Exponential Generating Functions

$$(a_0, a_1, \dots) \rightsquigarrow \hat{A}(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

$$(b_0, b_1, \dots) \rightsquigarrow \hat{B}(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} \underbrace{\left(\sum_{k=0}^{n} \frac{a_k}{k!} \cdot \frac{b_{n-k}}{(n-k)!}\right) x^n}_{\underbrace{\left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right)}_{= c_n} \frac{x^n}{n!}} = \hat{C}(x) = \hat{A}(x) \cdot \hat{B}(x)$$

<u>Thm</u>: (Product formula for exponential generating functions)

$a_n :=$ # of ways to build a structure of Type I on $[n]$

$b_n :=$ — " — II — " —

$\Rightarrow c_n :=$ # ways to do the following:

1. Partition $[n]$ into two subset $A_1, A_2$, $A_1 \cup A_2 = [n]$
2. Build structure Type I on $A_1$
3. — " — II on $A_2$

Then $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$

and $\hat{C}(x) = \hat{A}(x)\, \hat{B}(x)$

Example: Football coach, $n$ players

- Divide them into two groups $A_1$, $A_2$
- Each player in $A_1$ should take R, W, or G shirt and then form a line
- Players in $A_2$ should take a B shirt and then form a line

How many ways can this be done? $c_n$

$$a_k = \underbrace{3^k}_{\text{\# of ways to distribute shirts}} \underbrace{k!}_{\text{\# of ways to form a line}}$$

$$b_k = k!$$

$$c_n = \sum_{k=0}^{n} \binom{n}{k} \cdot a_k b_{n-k}$$

$$\hat{A}(x) = \sum_{k=0}^{\infty} k!\, 3^k \frac{x^k}{k!} = \sum_{k=0}^{\infty} (3x)^k = \frac{1}{1-3x}$$

$$\hat{B}(x) = \sum_{k=0}^{\infty} k!\, \frac{x^k}{k!} = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\hat{C}(x) = \hat{A}(x)\,\hat{B}(x) = \frac{1}{1-3x} \cdot \frac{1}{1-x} = \frac{A^{=\frac{3}{2}}}{1-3x} + \frac{B^{=-\frac{1}{2}}}{1-x}$$

$$= \frac{3}{2} \sum_{n=0}^{\infty} (3x)^n - \frac{1}{2} \sum_{n=0}^{\infty} x^n = \sum \left(\frac{3}{2} 3^n - \frac{1}{2}\right) x^n$$

$A + B = 1$

$-A - 3B = 0$

$B - 1 - 3B = 0$

$-1 = 2B$

$$\Rightarrow c_n = \frac{3^{n+1} - 1}{2}\, n!$$

## Derivative of exponential generating fns.

$$\left(\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}\right)' = \sum_{n=1}^{\infty} a_n \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!}$$

**Example:** Bell numbers $B(n) = \#$ of partitions of $[n]$ (into non-empty parts; $B(n) = \sum_{i=0}^{n} S(n,i)$)

$$B(n+1) = \sum_{i=0}^{n} \binom{n}{i} \cdot B(i)$$

Classify according to size $(n-i)$ of the part of element $n+1$

# ways to choose the elements NOT in the same part as $n+1$

# of ways to partition the rest (the elements NOT in the same part as $n+1$)

**Product Formula**

$a_i = B(i) \longrightarrow \hat{\alpha}(x) = \sum B(i) \frac{x^i}{i!}$

$b_i = 1 \qquad \hat{\beta}(x) = \sum \frac{x^i}{i!} = e^x$

$$c_n = \sum_{i=0}^{n} \binom{n}{i} B(i) = B(n+1) \rightsquigarrow \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} B(n+1)\frac{x^n}{n!} = \left(\sum_{n=0}^{\infty} B(n)\frac{x^n}{n!}\right)'$$

$$\hat{\alpha}(x) \cdot e^x = \hat{\alpha}(x) \cdot \hat{\beta}(x) = \left(\hat{\alpha}(x)\right)'$$

$$e^x = \frac{\alpha(x)'}{\alpha(x)} = \left(\ln \hat{\alpha}(x)\right)'$$

$$e^x + C = \ln \hat{\alpha}(x)$$

$x = 0 \rightsquigarrow \ln \alpha(0) = 0 \Rightarrow C = -1$

$$\boxed{e^{e^x - 1} = \hat{\alpha}(x)}$$

~~Prop~~ Prop (k-wise Product Formula for Exponential Generating Functions)

~~Recall Setting~~ For $j=1,2,\dots,k$, let $(a_0^{(j)}, a_1^{(j)}, \dots)$ sequence such that $a_n^{(j)} := \#$ of ways to build Structure of Type $j$ on $[n]$.

Let $\hat{A}_j(x) := \sum_{n=0}^{\infty} a_n^{(j)} \frac{x^n}{n!}$

Then $\prod_{j=1}^{k} \hat{A}_j(x)$ is the (exponential) generating function for the sequence

$d_n := \#$ of ways to do the following (maybe empty)

① ~~Split~~ Partition $[n]$ into $k$ subsets $S_1, S_2, \dots, S_k$

② Build structure of Type $j$ on $S_j$

Pf: Induction on $k$

- $k=1, 2$ ✓
- Do the procedure as follows:
  1. Partition $[n]$ into two subsets $S^*$ and $S_k$
  2. Build the "$(k-1)$ subset"-structure on $S^*$
  3. Build Structure Type $k$ on $S_k$

~~Generating~~ Exp. Generating function for the counting sequence of the procedure in ② is $\prod_{j=1}^{k-1} \hat{A}_j(x)$ (by Induction)

Exponential generating function for the counting sequence in ③ is $\hat{A}_k(x)$

So by the (pairwise) Product Formula, the exponential generating function for the whole process is

$$\boxed{\prod_{j=1}^{k} \hat{A}_j(x)}$$

# Composition of Exponential Generating Functions

Recall setup: $(a_n), (b_n)$, Structure of Type I, II, $\hat{A}(x), \hat{B}(x)$

$$\boxed{\text{What do the coefficients of } \hat{B}(\hat{A}(x)) \text{ represent?}}$$

Again to make sense to $\hat{B}(\hat{A}(x))$ we require $\boxed{a_0 = 0}$

Special Case: $b_n = 1 \ \forall n \geq 0$, that is $\hat{B}(x) = e^x$

Prop: $e^{A(x)}$ is the e.g.f. of the sequence

$h_n :=$ # of ways to do the following:

(1.) Partition $[n]$ into nonempty subsets (unspecified number)

Define $\boxed{h_0 := 0}$ (2.) Build a structure of Type I on each

Pf: Classify according the number of nonempty subsets in the partition

$h_n^{(k)} :=$ # of choices with $k$ non-empty subsets in partition

Then • Sum Rule $\Rightarrow$ $\boxed{h_n = \sum_{k=1}^{n} h_n^{(k)} = \sum_{k=1}^{\infty} h_n^{(k)}}$ since $h_n^{(k)} = 0 \ \forall k > n$

• Exponential Generating function: $\sum_{k=0}^{\infty} h_n^{(k)} \frac{x^{(k)}}{n!} = \frac{1}{k!}(\hat{A}(x))^k$

(1.) Product Formula also counts partitions with (some) empty sets: these have no contribution here, since $a_0 = 0$ ✓
(2.) Product Formula must be divided by $k!$, since here sets of partition are not ~~labeled~~

$$\hat{H}(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!} = h_0 + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} h_n^{(k)} \frac{x^n}{n!} = 1 + \sum_{k=1}^{\infty} \underbrace{\sum_{n=1}^{\infty} h_n^{(k)} \frac{x^n}{n!}}_{\frac{(\hat{A}(x))^k}{k!}} = e^{\hat{A}(x)}$$

Example: Arrange $n$ people into groups and seat them around circular table.

Sol: $a_k = (k-1)! \ \forall k \geq 1$
$a_0 := 0$

$$\rightsquigarrow \hat{A}(x) = \sum_{k=0}^{\infty} (k-1)! \frac{x^k}{k!} = \sum_{k=1}^{\infty} \frac{x^k}{k} = \ln \frac{1}{1-x}$$

$b_k = 1 \ \forall k \geq 0$

$$\Rightarrow \hat{G}(x) = e^{\ln\left(\frac{1}{1-x}\right)} = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} \underbrace{k!}_{g_k} \frac{x^k}{k!}$$

Real Time Exercise: Combinatorial solution?

Example: # of partition of $[n]$ into block of sizes 3, 4, or 9

$$\overset{\|}{f_n} \qquad \hat{F}(x) := \sum_{k=0}^{\infty} f_k \frac{x^k}{k!}$$

$$a_k := \begin{cases} 1 & k=3 \\ 0 & k \neq 3 \end{cases} \qquad b_k := \begin{cases} 1 & k=4 \\ 0 & k \neq 4 \end{cases} \qquad c_k := \begin{cases} 1 & k=9 \\ 0 & k \neq 9 \end{cases}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\hat{A}(x) = \frac{x^3}{3!} \qquad \hat{B}(x) = \frac{x^4}{4!} \qquad \hat{C}(x) = \frac{x^9}{9!}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$e^{\hat{A}(x)}$ e.g.f. of # of partitioning into parts of size 3, (Choice of $a_k$ makes sure that in a partition of $[n]$ into subsets only those are counted where each part is of size 3)

$e^{\hat{B}(x)}$

$e^{\hat{C}(x)}$

$e^{\hat{A}(x)+\hat{B}(x)+\hat{C}(x)}$ is e.g.f. of partitioning into three parts: first to be partitioned into 3s, second to be partitioned into 4s, third to be partitioned into 9s

# Composition of Exponential Generating Fns

$$(a_0, a_1, \ldots) \longrightarrow \hat{A}(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

$$(b_0, b_1, \ldots) \longrightarrow \hat{B}(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} c_n \frac{x^n}{n!} = \sum_{k=0}^{\infty} b_k \underbrace{\frac{\hat{A}(x)^k}{k!}}_{\text{Makes sense only if } \boxed{a_0 = 0}} = \hat{B}(\hat{A}(x))$$

Thm: (Composition Formula f Exponential Generating Fn)

$a_n :=$ # of ways to build a structur of Type I on $[n]$

$b_n :=$ # — " — Type II on $[n]$

$\Rightarrow$ $c_n :=$ # ways to do the following:

1. Partition $[n]$ into nonempty subsets (unspecified number)
2. Build structure of Type I on each subset
3. Build — " — Type II on <u>set</u> of subsets

$$\hat{C}(x) = \sum c_n \frac{x^n}{n!} = \hat{B}(\hat{A}(x))$$

Pf: classify according to # of parts in partition:

$c_n^{(k)}$ = # of ways to do above with <u>k</u> parts (nonempty) $\Rightarrow c_n = \sum_{k=1}^{n} c_n^{(k)}$

$$\hat{C} = \sum_{k=0}^{\infty} \hat{C}^{(k)}(x) \qquad \hat{C}^{(k)}(x) = \underbrace{b_k}_{\text{put structure of Type II on } [k]} \cdot \underbrace{\hat{A}^k(x)}_{\text{Product formula for k parts}} \cdot \frac{1}{k!}$$

Example: $n$ distinct cards

- Split them into (non-empty) decks, each with an even number of cards
- order each deck
- order decks in a line

How many ways can we do this? $g_n$

Composition Formula:

$$a_n := \begin{cases} n! & n \text{ is even } \geq 2 \\ 0 & n \text{ is odd or } n=0 \end{cases}$$

$$b_n := n! \quad \forall n \geq 0$$

$$\hat{A}(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \sum_{\substack{n \geq 2 \\ n \text{ is even}}} x^n = \frac{x^2}{1-x^2}$$

$$\hat{B}(x) = \sum b_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\Rightarrow \hat{G}(x) = \hat{B}(\hat{A}(x)) = \frac{1}{1 - \frac{x^2}{1-x^2}} = \frac{1-x^2}{1-2x^2} =$$

$$= 1 + \frac{x^2}{1-2x^2} = 1 + x^2 \sum_{m=0}^{\infty} (2x^2)^m = 1 + \sum_{m=0}^{\infty} 2^m x^{2m+2}$$

$$\Rightarrow \quad g_{2m+1} = 0 \quad \forall m$$

$$g_{2m} = 2^{m-1}(2m)! \quad \forall m$$

$$\Rightarrow \quad g_n = 2^{\frac{n}{2}-1} \cdot n!$$