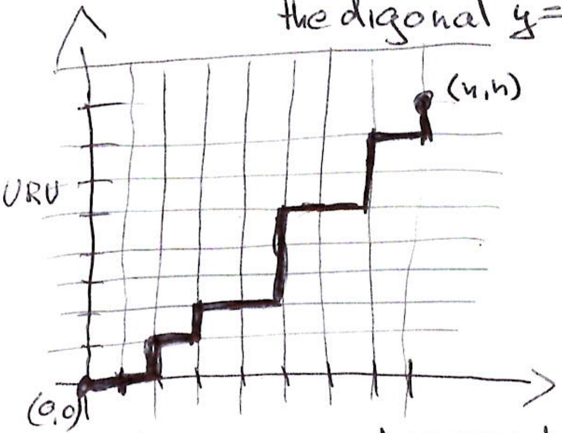
Catalan numbers – alternative solution

First idea: Transfer the problem to counting latticepaths:
(small)

$$\left\{(b_1,\dots,b_{2n})\in\{-1,1\}^{2n} : \sum_{i=1}^{2n} b_i = 0,\ \forall j\ \sum_{i=1}^{j} b_i \geq 0\right\} \longleftrightarrow \left\{\begin{array}{l}\text{R/U lattice paths} \\ \text{from } (0,0) \text{ to } (n,n) \\ \text{NOT "going over"} \\ \text{the diagonal } y=x\end{array}\right\}$$

Bijection

$(1,1,-1,1,-1,1,1,-1,-1,-1,1,1,-1,-1,1,-1) \longleftrightarrow$ RRURURRUUURRUURU

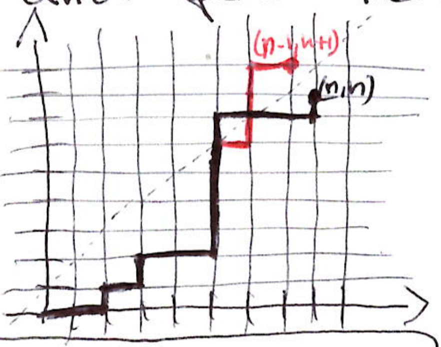


Second (small) idea: Count exactly those path we are NOT interested in and subtract from "everything" (i.e. from $\binom{2n}{n}$)

Let Q_n the set of those $(0,0),(n,n)$ paths that DO CROSS the diagonal $y=x$
Every path in Q_n reaches the line $y=x+1$.

Third (great!) idea: Make the following transformation of $P\in Q_n$

- Find the first point $(k,k+1)$ where P hits $y=x+1$ and reflect the rest of P about $y=x+1$ to create $\widehat{P}$.



$\widehat{P}$ is a lattice path from $(0,0)$ to $(n-1,n+1)$

$$\widehat{Q}_n := \{\widehat{P} : P\in Q_n\}$$

$$Q_n \xleftrightarrow[\text{bijection}]{} \widehat{Q}_n = \{\text{ALL lattice paths from } (0,0) \text{ to } (n-1,n+1)\}$$

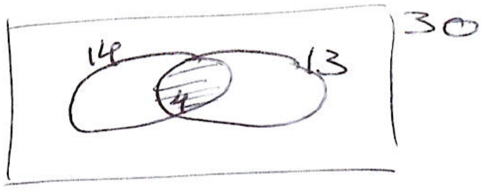
One can do the operation backward for $\forall$ such lattice path and get a lattice path from $(0,0)$ to (n,n) crossing $y=x+1$.

$$C_n = \binom{2n}{n} - |Q_n| = \binom{2n}{n} - |\widehat{Q}_n| =$$
$$= \binom{2n}{n} - \binom{n+1+n-1}{n-1} =$$
$$= \binom{2n}{n} - \binom{2n}{n-1} = \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}$$

Inclusion - Exclusion

In a village there is opportunity to play either soccer or basketball in a club. In the 8^{th} grade there are 30 students, 14 plays soccer and 13 basketball. How many plays neither sports?

Not enough information!



How many play both? 4

Then $\underbrace{30 - 13 - 14}_{\text{students playing both were subtracted twice}} + 4 = \underline{7}$ students do not play any sports.

General ~~phenomene~~ situation:

In a base set S the elements could have n propties $P_1, \dots, P_n$; (an element might have ~~many~~ more than one)

We WANT TO KNOW: • How many elements have none of the properties?

(or • How many elements have AT LEAST ONE of the properties?)

Example: How many integers in $[30]$ are relatively prime to 30?

$$S = [30]$$

$$(a,30)=1 \iff 2 \nmid a \text{ and } 3 \nmid a \text{ and } 5 \nmid a$$

$A_j := \{x \in [30] : j \mid x\}$ ⇝ integers up to divisible by j

We are interested in

$$\boxed{R = S \setminus A_2 \cup A_3 \cup A_5 = \overline{A_2} \cap \overline{A_3} \cap \overline{A_5}}$$

$$|A_2 \cup A_3 \cup A_5| = \underbrace{|A_2| + |A_3| + |A_5|}_{\text{elements appearing in at least two sets are counted twice}} \underbrace{- |A_2 \cap A_3| - |A_3 \cap A_5| - |A_2 \cap A_5|}_{\text{elements appearing in exactly two are now counted once ✓ but elements appearing in all three were added 3 times and then subtracted 3 times}}$$

$$+ |A_2 \cap A_3 \cap A_5| \quad \text{NOW OK ✓}$$

- $|A_j|$ = # integers divisible by j up to 30 $= \left\lfloor \frac{30}{j} \right\rfloor$ (every jth integer is divisible by j)
- p, q are primes $\implies A_p \cap A_q = A_{pq}$ ($p \mid x$ AND $q \mid x \iff pq \mid x$) (since x has unique factorization into primes)

$$\implies |A_2 \cup A_3 \cup A_5| = 15 + 10 + 6 - 5 - 3 - 2 + 1 = 22 \implies |R| = 30 - 22 = \underline{\underline{8}}$$

General Inclusion-Exclusion Formula

Thm: S finite set, $A_1, \dots, A_n \subseteq S$ subsets,

$$\Rightarrow \left| S \setminus \bigcup_{i=1}^{n} A_i \right| = \sum_{I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|$$

Corollary: $A_1, \dots, A_n$ are finite sets

$$|A_1 \cup \dots \cup A_n| = \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|$$

Pf of Corollary: Choose $\boxed{S = A_1 \cup \dots \cup A_n}$

$\bigcap_{i \in \emptyset} A_i = S$ (set of those elements in S which are present in EVERY ~~element~~ set A_i with an index in $\emptyset$)

$$0 = \left| S - \bigcup_{i=1}^{n} A_i \right| = |S| + \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|$$

Pf of Thm: many: • induction on n $\quad n=0$ $|S| = \left|\bigcap_{i \in \emptyset} A_i\right|$

$n=1$ $|S - A_1| = |S| - |A_1|$

$\vdots$

• checking how many times an element is counted on both sides

Pf of Thm: using polynomial identity

$$\prod_{i=1}^{n}(1+x_i) = \sum_{I\subseteq[n]} \prod_{i\in I} x_i$$

Let $f_i : S \longrightarrow \{0,1\}$ be the characteristic fn of A_i

$$f_i(a) := \begin{cases} 1 & \text{if } a\in A_i \\ 0 & \text{if } a\notin A_i \end{cases}$$

So

$$\left|S - \bigcup_{i=1}^{n} A_i\right| = \sum_{a\in S} \underbrace{\prod_{i=1}^{n} \underbrace{(1-f_i(a))}_{\overset{\shortparallel}{1} \text{ iff } a\notin A_i}}_{\overset{\shortparallel}{1} \text{ iff } a\notin A_i\ \forall i=1,\dots,n, \text{ (otherwise } 0)}$$

$$= \sum_{a\in S} \sum_{I\subseteq[n]} \prod_{i\in I}(-f_i(a))$$

$$= \sum_{I\subseteq[n]} (-1)^{|I|} \sum_{a\in S} \underbrace{\prod_{i\in I} f_i(a)}_{\overset{\shortparallel}{1} \Leftrightarrow a\in\bigcap_{i\in I} A_i \text{ (otherwise } 0)}$$

$$= \sum_{I\subseteq[n]} (-1)^{|I|} \left|\bigcap_{i\in I} A_i\right|$$

$\square$

Application: Euler totient function $\varphi(n) = |\{d \in [n] : \gcd(d,n) = 1\}|$

A formula? For $n = p$ prime $\rightsquigarrow \varphi(p) = p - 1$

For $n = p^\alpha$ prime power $\rightsquigarrow \varphi(p^\alpha) = p^\alpha - p^{\alpha-1}$

($\{p, 2p, \dots, p^{\alpha-1} \cdot p\}$ are the NON-relative primes)

In general: Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ with $p_1, \dots, p_k$ distinct, $\alpha_1, \dots, \alpha_k \geq 1$

$$\boxed{\varphi(n)} = |\{d \in [n] : \gcd(d,n) = 1\}| = \left|[n] \setminus \bigcup_{i=1}^{k} A_{p_i}\right|$$

For $\gcd(d,n) = 1$ d and n should have no common prime factor

where $A_j = \{\ell \in \mathbb{N} : j \mid \ell\}$

Inclusion-Exclusion $\Longrightarrow$

$$= \sum_{S \subseteq [k]} (-1)^{|S|} \left|\bigcap_{i \in S} A_{p_i}\right|$$

Claim: $\gcd(a,b) = 1 \Rightarrow A_a \cap A_b = A_{ab}$

Pf: $a \mid c$ AND $b \mid c \iff ab \mid c$

(consider prime factorization of a, b, c)

Corollary: $\bigcap_{i \in S} A_{p_i} = A_{\prod_{i \in S} p_i}$

Pf: By induction on $|S|$. $(\gcd(p_i, \prod_{j \in S, j \neq i} p_j))$

$$= \sum_{S \subseteq [k]} (-1)^{|S|} \left|A_{\prod_{i \in S} p_i}\right|$$

$$= \sum_{S \subseteq [k]} (-1)^{|S|} \left\lfloor \frac{n}{\prod_{i \in S} p_i} \right\rfloor$$

$$= \sum_{S \subseteq [k]} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i}$$

$$= n \sum_{S \subseteq [k]} \prod_{i \in S} \left(\frac{-1}{p_i}\right)$$

$$\boxed{= n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)}$$

Claim $\forall j \quad |A_j| = \left\lfloor \frac{n}{j} \right\rfloor$

Pf: $\forall j^{\text{th}}$ integer is in A_j $\square$

Application: # of derangements

Def: $\boxed{\text{derangement}}$: permutation with no fixed point

$\boxed{\pi \in S_n \text{ s.t. } \forall i \in [n]\ \pi(i) \neq i}$ (every cycle has length at least $\underline{2}$)

$D(n) =$ # of derangements over $[n]$ ~ "Hat-check lady Problem"

Let $A_i = \{\pi \in S_n : \pi(i) = i\} \subseteq S_n$

WANT: $D(n) = \left| S_n \setminus \bigcup_{i=1}^{n} A_i \right| =$

$$= \sum_{S \subseteq [n]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|$$

$$= \sum_{S \subseteq [n]} (-1)^{|S|} \underbrace{\left| \{\pi \in S_n : \forall i \in S\ \pi(i) = i\} \right|}_{=(n-|S|)!}$$

(since values on $|S|$ elements are fixed, rest is free)

$$= \sum_{k=0}^{n} \sum_{S \in \binom{[n]}{k}} (-1)^k (n-k)! = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)! = \boxed{n! \sum \frac{(-1)^k}{k!}}$$

Corollary: $\dfrac{D(n)}{S_n} = \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \xrightarrow[n \to \infty]{} \frac{1}{e}$

Corollary: $D(n) = \left\| \frac{n!}{e} \right\|$ ($\|z\|$ is the closest integer to z)

Pf: $\left| \frac{n!}{e} - D(n) \right| = \left| n! \sum_{k=n+1}^{\infty} (-1)^k \frac{1}{k!} \right| = \left| \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \dots \right| < \frac{1}{n+1} < \frac{1}{2}$

$\square$