Pigeonhole Principle

Question: Are there two people in Berlin with the SAME number of strands of hair?

YES! No person has more than a million (1-200,000 on average) and Berlin has 3.5 million inhabitants.

(If the answer was NO, then every hairstrand-number would have at most one representative in Berlin and the Berlin could have at most 1,000,000 residents, a contradiction...)

Pigeonhole Principle (Schubfach Prinzip)

Informal: when $n+1$ pigeons sit in n pigeonholes then $\exists$ a pigeonhole with at least 2 pigeons

Formal: X, Y finite sets $|X| > |Y|$

$$\forall \underbrace{f : X}_{\text{pigeons}} \to \underbrace{Y}_{\text{pigeonholes}} \quad \exists\, x_1, x_2 \in X,\ x_1 \neq x_2,\ f(x_1) = f(x_2)$$

Simplest Examples:

- Among three ordinary people there are two that have the same sex. (Kleitman)
- Among 13 people there are two who were born in the same month.

Etc....

(Less and less obvious) Applications

Claim:

- $\forall B \in \binom{[200]}{101} \quad \exists x, y \in B$ s.t. $x \mid y$

First thoughts: Statement is best possible.
(same is false with 100-element subsets instead of 101 element subsets.)

Pf: How to create pigeonholes?
Write $\forall x \in [200]$ as $x = 2^{s_x} \cdot b_x$ with $s_x \in \mathbb{N}_0$, b_x odd

function $x \longmapsto b_x$

$\quad\;\; \cap \qquad\quad \cap$

$\quad\; B \qquad \{y \in [200] : y \text{ is odd}\} = Y$

Pigeons $\qquad$ pigeonholes

$|Y| < |B|$ (strictly more pigeons than pigeonholes)

$\;\,\| \qquad \|$

$100 \quad 101 \quad \Rightarrow x_1 \neq x_2 \in B$ with $b_{x_1} = b_{x_2}$. Say $x_1 < x_2$

$\Rightarrow \boxed{x_1} = 2^{s_{x_1}} \cdot b_{x_1} \mid 2^{s_{x_2}} \cdot b_{x_2} = \boxed{x_2}$ since $\frac{x_2}{x_1} = 2^{s_{x_2} - s_{x_1}} \in \mathbb{N}$

Chinese Remainder Theorem

$\forall m, n \in \mathbb{N}$ $\quad gcd(m,n) = 1$

$\forall a, b \in \mathbb{N}_0$ with $0 \le a < m$

$0 \le b < n$

$\exists x \in \mathbb{Z}$ s.t. $\quad x \equiv a \pmod{m}$ $\quad$ [i.e. $\exists q \in \mathbb{Z}: x = m \cdot q + a$]

AND

$x \equiv b \pmod{n}$ $\quad$ [i.e. $\exists r \in \mathbb{Z}: x = n \cdot r + b$]

<u>Example</u>: Is there an $x \in \mathbb{N}$ s.t. $\quad x \equiv 26 \pmod{63}$

AND

$x \equiv 13 \pmod{40}$

YES! (since $gcd(63, 40) = 1$) [smallest such positiv $x = 1853$]

<u>Proof</u>: Let $A = \{a, m+a, 2m+a, \ldots, (n-1)m + a\}$

$\boxed{|A| = n}$ $\quad$ EVERY element of A is $\equiv a \pmod{m}$

<u>Case1</u>: $\exists x \in A$ s.t. $x \equiv b \pmod{n}$ ✓

<u>Case2</u>: $\forall x \in A$ $\quad x \not\equiv b \pmod{n}$

$$\left.\begin{array}{l} f: x \longmapsto \text{remainder} \pmod{n} \\ \quad\; \cap \\ \quad\; A \longrightarrow \{0, 1, 2, \ldots, n-1\} \setminus \{b\} \\ n \text{ pigeons into } n-1 \text{ pigeonholes} \end{array}\right\} \Longrightarrow \begin{array}{l} \exists x_1, x_2 \in A \quad x_1 \neq x_2 \\ x_1 \equiv x_2 \pmod{n} \end{array}$$

$$\Longrightarrow \left.\begin{array}{l} \exists\, i_1 < i_2, \; i_1, i_2 \in \{0, 1, \ldots, n-1\} \\ i_1 m + a \equiv i_2 m + a \pmod{n} \end{array}\right\} \Rightarrow (i_2 - i_1) \cdot m \equiv 0 \pmod{n} \Rightarrow n \mid (i_2 - i_1) m \xRightarrow{\text{since } gcd(m,n)=1} n \mid i_2 - i_1$$

since $|i_2 - i_1| < n$ $\Rightarrow i_2 = i_1$

PP – General Form

Let $q_1, \dots, q_n \in \mathbb{N}$

When $1 + \sum_{i=1}^{n}(q_i - 1)$ objects are placed in n (numbered) boxes then $\exists\, i \in [n]$ s.t. Box i contains at least q_i objects

Pf: By contradiction: If the statement were not true THEN $\forall i \in [n]$ Box i would contain LESS Than q_i objects

$\Rightarrow$ ALLTOGETHER AT MOST $\sum_{i=1}^{n} q_i - 1$ objects are in the boxes

Remark: Pf is so simple that usually it is shorter to just write it out fully rather than fully explain how to apply (and describe pigeons and pigeonholes)

Special Case (Averaging)

$q_1 = \dots = q_n = q \in \mathbb{N}$

When n Boxes together hold MORE than $n(q-1)$ objects then $\exists$ a Box with q objects.

Reformulation

When n Boxes together hold Q objects

THEN (1) $\exists$ a Box with AT LEAST $\left\lceil \frac{Q}{n} \right\rceil$ objects

(2) $\exists$ a Box with AT MOST $\left\lfloor \frac{Q}{n} \right\rfloor$ objects

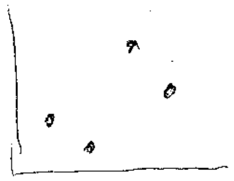
Remarks: The Pigeonhole Principle is only an EXISTENCE proof technique

There is no general method (algorithm) which FINDS the pigeonhole with the desired properties FAST

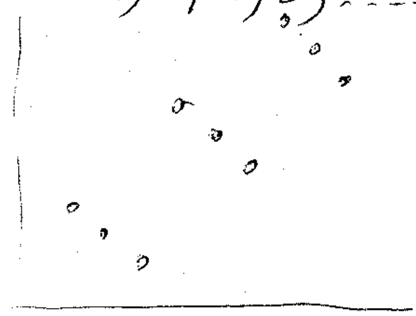
(Even though we KNOW it exists...)

Question: How long monotone subsequence are we guaranteed to have in a sequence of length n?

(Realtime) Exercise: $n = 2, 3, 4, 5, \dots, 9$ Constructions



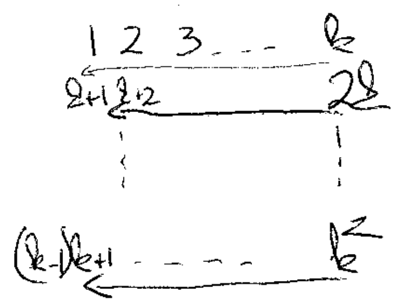
2 1 4 3



321 654 987

Remark: Equality of elements in the sequence only "helps" finding monotone subsequence (we don't require strictly monotone subsequence)

$\boxed{\text{Construction for } \underline{\text{arbitrary}}\ \underline{n = k^2}}$:



Read rows backwords one after the other.

$(k, k-1, \dots, 1, 2k, 2k-1, \dots, k+1, \dots, k^2, k^2-1, \dots (k-1)k+1)$

Claim Longest monotone subsequence is of length k

Pf: ~~this~~ $\boxed{\text{AT LEAST } k}$: First k elements is a monotone decreasing sequence

$\boxed{\text{AT MOST } k}$: $\forall$ ~~dec~~ increasing subsequence can have AT MOST ONE element from each row. → Length is at most k
There are k rows

$\forall$ decreasing subsequence can have at MOST ~~k elemen~~ ONE element from each column → Length is at most k
There are k columns

$\square$

We prove that this construction is BEST POSSIBLE in the following sense: $\forall$ sequence of just ONE MORE contains a LONGER monotone subsequence.

<u>Thm</u> (Erdős-Szekeres, 1930's)
$\forall$ sequence of k^2+1 reals contains a monotone subsequence of length $k+1$

<u>Pf</u>: ~~Classify~~ Let $x_1, x_2, \ldots, x_{k^2+1}$ be a sequence of length k^2+1.

<u>Classify</u> elements (pigeons) based on

$x_j \longrightarrow i_j :=$ length of longest INCREASING subsequence ENDING at x_j

<u>Case 1.</u>: $\exists$ index $l \in [k^2+1]$ $i_l \geq k+1$ DONE ✓

<u>Case 2</u>: $\forall$ index $j \in [k^2+1]$ $i_j \leq k$

$\exists$ at most k classes (pigeonholes) $I_p := \{x_j : i_j = p\}$ $1 \leq p \leq k$

covering $\{x_1, \ldots, x_{k^2+1}\} = \bigcup_{p=1}^{k} I_p$

By averaging $\exists$ class I_r, $|I_r| \geq \left\lceil \frac{k^2+1}{k+1} \right\rceil \geq k+1$

Elements of I_r form a decreasing sequence (of length $k+1$). Indeed $\forall$ $x_l, x_j \in I_r$, $l < j$,

$\boxed{x_l \geq x_j}$, otherwise $r = i_l < i_j = r$ $\lightning$

PP $\Rightarrow$ $\forall$ coloring $2k-1$ points with 2 colors contains k with same color

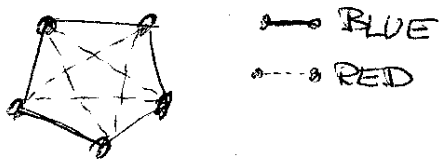
This best possible: $2k-2$ would not be enough for the same conclusion.

<u>Question</u>: What if we color 2-element subsets (instead of the 1-element subsets)?

How many points do we need to guarantee k points such that ALL PAIRS from them colored with the same color?

- Or just 3 points? (Let's be modest...)
- Is there such a (large?) number?

Are 5 points enough? NO!



(5 was enough when we wanted 3 POINTS WITH THE same color)

6 is enough!

<u>Proposition</u>: $\forall c: \binom{[6]}{2} \to \{R, B\}$ $\exists$ monochromatic $M \subseteq [6]$, $|M| = 3$

<u>Def</u>: Given a set V with all its 2-element subsets colored with Red or Blue, a subset $K \subseteq V$ is called $\boxed{\text{monochromatic}}$ $\boxed{\text{(m.c.)}}$ if all two-element subsets of K have the same color.

Pf of Proposition: Let $c : \binom{[6]}{2} \to \{R, B\}$ be a 2-coloring.

- Look at the pairs containing 1. There are 5:
  12, 13, 14, 15, 16.

  By PP at LEAST $\left\lceil \frac{5}{2} \right\rceil = 3$ of them have the same color

- Say for $2 \le i < j < k \le 6$ $c(1i) = c(1j) = c(1k) = R$.

Case 1.: $\exists$ a pair among ij, jk, ik, that is colored R

$\Rightarrow$ Done! Indeed: say $c(ij) = R$

$\Rightarrow \{1, i, j\}$ m.c. 3-set

Case 2: $\forall$ pair among ij, jk, ik is not R

$\Rightarrow c(ij) = c(jk) = c(ik) = B$

$\Rightarrow \{i, j, k\}$ is a m.c. B subset of size 3.

$\square$

How about finding a m.c. ~~k~~-set?

Is there such a (large?) integer that guarantees that?

- 18 points is enough, 17 is not!

Def: (Symmetric) Ramsey number $R(k) := \min\{N : \forall c : \binom{[N]}{2} \to \{R, B\}\ \exists \text{ m.c. } K \subseteq [N], |K| = k\}$

Example: $R(2) = 2$ ✓

$R(3) = 6$ ~~…~~ $\leq$ by Proposition

$\geq$ construction of 2-coloring of $\binom{[5]}{2}$ without m.c. 3-set.

Ramsey's Thm: $R(k) < 4^k$ (In particular $R(k) < \infty$)

Pf: ~~…~~ For an UPPER BOUND on the Ramsey-number we need to ~~…~~ take an arbitrary 2-coloring ~~of~~ $c : \binom{[N]}{2} \to \{R, B\}$, ~~…~~ and somehow FIND a subset $K \subseteq [N], |K| = k$ which is m.c.

Let $N = 2^{2k-1}$. Let $c : \binom{[N]}{2} \to \{\text{Red}, \text{Blue}\}$ arbitrary.

Define $i_1 := 1$, $V_1 := [N]$. ~~…~~ Let $\begin{cases} R_1 := \{j \in V_1 : c(i_1 j) = \text{Red}\} \\ B_1 := \{j \in V_1 : c(i_1 j) = \text{Blue}\} \end{cases}$

By averaging either $|R_1|$ or $|B_1|$ is $\geq \left\lceil \frac{|R_1| + |B_1|}{2} \right\rceil = \left\lceil \frac{2^{2k-1} - 1}{2} \right\rceil = 2^{2k-2}$

Let V_2 be this (larger) set. So $|V_2| \geq 2^{2k-2}$

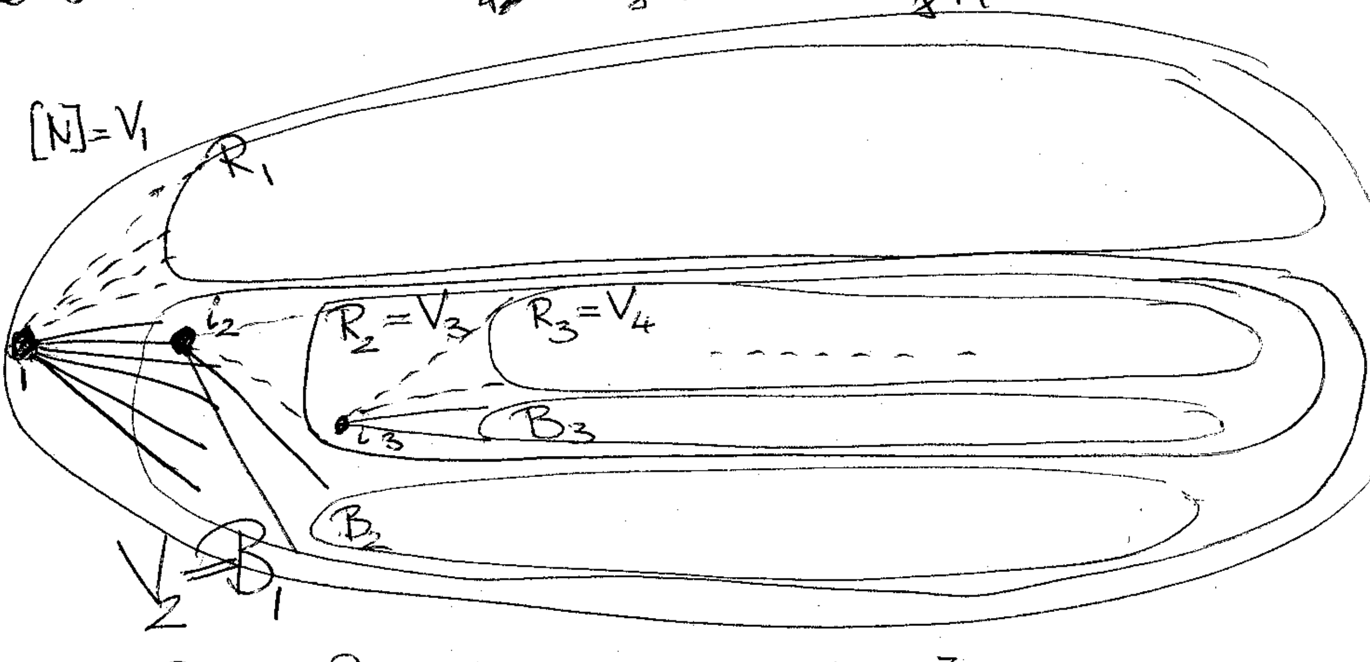
Important: All pairs $1j$, $j \in V_2$, have the SAME color
Call this color C_1 ($\in \{\text{Red}, \text{Blue}\}$)

For $\forall j = 1, 2, \dots, 2k-1, 2k$
inductively construct elements $1 := i_1 < i_2 < \dots < i_{2k-1} < i_{2k}$
and subsets $[N] := V_1 \supseteq V_2 \supseteq \dots \supseteq V_{2k-1} \supseteq V_{2k}$

such that $\forall j = 1, \dots, 2k$
- $i_j \in V_j$
- $|V_j| \geq 2^{2k-j}$

$\forall j = 1, \dots, 2k-1$
- $\exists$ color $C_j \in \{\text{Red}, \text{Blue}\}$ s.t. $\forall x \in V_{j+1}$ $c(i_j x) = C_j$

Suppose we have the construction up to index $j < 2k$, let's construct i_{j+1} and V_{j+1}:



Let $R_j := \{x \in V_j : c(i_j x) = \text{Red}\}$
$B_j := \{x \in V_j : c(i_j x) = \text{Blue}\}$

By averaging either $|B_j|$ or $|R_j|$ is $\geq \left\lceil \frac{|B_j| + |R_j|}{2} \right\rceil = \left\lceil \frac{|V_j| - 1}{2} \right\rceil \geq \left\lceil \frac{2^{2k-j} - 1}{2} \right\rceil = 2^{2k-j-1}$

Let V_{j+1} this (larger) set, so $|V_{j+1}| \geq 2^{2k-(j+1)}$

Define $i_{j+1} := \min\{x \in V_{j+1}\}$ and $C_{j+1} := c(i_j i_{j+1})$

So; we have a sequence of 2k-1 colors

$$C_1, C_2, ----, C_{2k-1} \in \{\text{Red}, \text{Blue}\}$$

By PP at least $\lceil \frac{2k-1}{2} \rceil$ k of them are the same

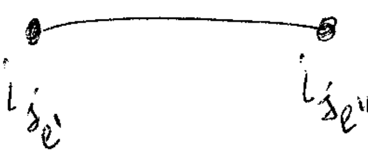
Say, for $1 \le j_1 < j_2 < \dots < j_k \le 2k-1$

$$C_{j_1} = C_{j_2} = \dots = C_{j_k} = \text{Red}$$

Then $i_{j_1} < i_{j_2} < \dots < i_{j_k} < i_{\overset{j_{k+1}}{2k}}$ is a

m.c. (k+1)-element set in Red

<u>WHY?</u> for $l' < l'' \le k+1$



is Red since $i_{j_{l''}} \in V_{j_{l''}} \subseteq V_{j_{l'+1}}$

and $\forall x \in V_{j_{l'+1}}$ $c(i_{j_{l'}}, x) = C_{j_{l'}} = \text{Red}$

So we found a (k+1)-set which is m.c.

$$\Longrightarrow 2^{2k-1} = N \ge R(k+1) \qquad \square$$

<u>Remark:</u> In fact we have shown

$$R(k) \le 2^{2(k-1)-1} = \frac{4^k}{8}$$

$R(4) = 18$

$43 \le R(5) \le 49$

$102 \le R(6) \le 165$

$205 \le R(7) \le 540$

$\vdots$

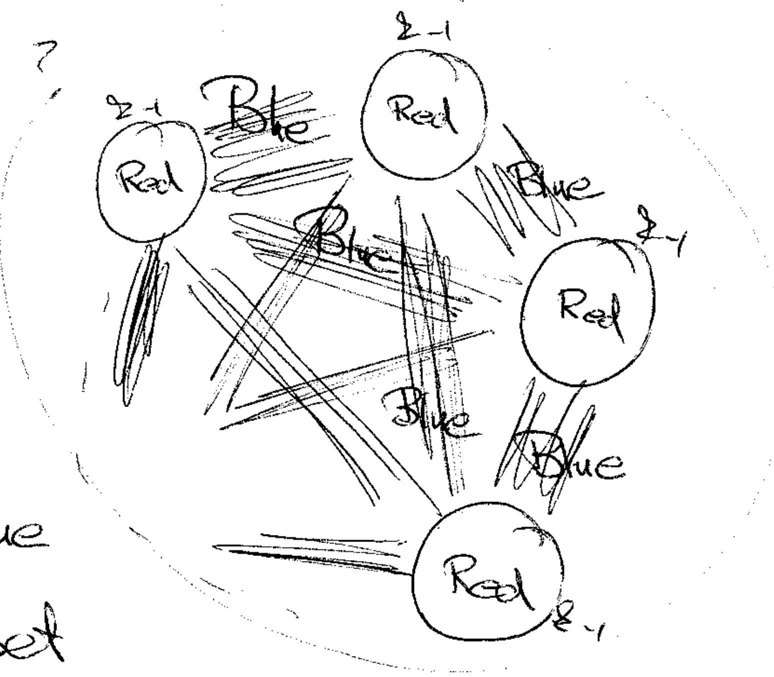
$798 \le R(10) \le 23536$

How good is 4^k as an upper bound?

For LOWER BOUND on $R(k)$ we want a CONSTRUCTION of a coloring of $\binom{[N]}{2}$ with Red/Blue such that $\nexists$ m.c. set of size k.

(and N is as large as possible... at least for a good lower bound)

ANY Construction?



$k-1$ sets U_i of size $k-1$

Pairs inside U_i: Red

between U_i U_j: Blue

No m.c. k-set

$$\Longrightarrow (k-1)^2 < R(k) < 4^k$$

BIG GAP!

Thm (Erdös 1947) $R(k) > \sqrt{2}^k$

Remarks
- Exponential lower bound! (as opposed to quadratic)
- Non-~~cons~~explicit construction: only EXISTENCE is proved
- Start of the Probabilistic Method in Combinatorics

Pf: Idea: ~~Blue~~ Consider ALL 2-colorings of $\binom{[N]}{2}$

- Enumerate the "bad" ones: the ones which have a m.c. k-set
- "Hope" it is strictly less than all $(=2^{\binom{N}{2}})$

How to count bad colorings?

Fix k-set $K \subseteq [N]$

Define $\mathcal{B}_K = \left\{ c: \binom{[N]}{2} \to \{\text{Red}, \text{Blue}\} : K \text{ is m.c. in } c \right\}$

set of bad colorings $\mathcal{B} = \bigcup_{K \in \binom{[N]}{k}} \mathcal{B}_K$

$\exists$ coloring with NO m.c. k-set if $|\mathcal{B}| < 2^{\binom{N}{2}}$

This is true if $\sum_{K \in \binom{[N]}{k}} |\mathcal{B}_K| = \binom{N}{k} \cdot 2^{\binom{N}{2} - \binom{k}{2} + 1} < 2^{\binom{N}{2}}$

# of pairs not in the fixed k-set

Red or Blue m.c. the fixed k-set

True if $\binom{N}{k} < 2^{\binom{k}{2} - 1}$

$\Uparrow$

$\left(\frac{Ne}{k}\right)^k < 2^{\frac{k-1}{2} k - 1}$

$\Updownarrow$

$N < 2^{\frac{k-1}{2}} \cdot k \cdot \frac{1}{e} \cdot \frac{1}{2^{1/k}}$ So $R(k) > (1+o(1)) \frac{\sqrt{2}}{e} \cdot k \cdot \sqrt{2}^k$

$\square$

Stronger upper bound (HW3)

Def: Ramsey number $R(k,l)$

Real time Exercise: $R(1,k) = R(k,1) = 1$

$R(2,k) = R(k,2) = k$

Thm (Erdős-Szekeres)

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

Pf: Induction on $k+l$

Corollary: $R(k) = R(k,k) \leq \binom{2k-2}{k-1} = \Theta\left(\frac{4^k}{\sqrt{k}}\right)$

Erdős \$500: Prove/disprove that $\lim_{k\to\infty} \sqrt[k]{R(k,k)}$ exists!

\$500: What is $\lim_{k\to\infty} \sqrt[k]{R(k,k)}$?

Nobody can show: $3.9999^k > R(k,k)$

or

$1.4143^k < R(k,k)$