~~The Distribution Problems~~

Partition Problems

n balls into k boxes — How?

n balls → numbered/indistinguishable

k boxes → numbered/indistinguishable

How? → $\forall$ box non-empty

$\forall$ box at most 1

~~$\forall$~~ arbitrary

· Distributing pennies to children

31 pennies – indistinguishable balls

3 children – numbered boxes

o o o o o | o o | o ___ o

A B C

Put down ~~31 pennies~~ 33 objects (31 pennies + 2 dividers) in a row $\binom{31+2}{2}$ (Choose positions of dividers)

In general: $\binom{n+k-1}{k-1}$

If all ~~boxes~~ children should get at least one penny: $\binom{n-k+k-1}{k-1}$

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k- multiset   Informal : repetition of elements allowed
$\{1,1,2,2,2,3\}$

Formal : non-decreasing sequence of k integers
$(1,1,2,2,2,3)$
$1 \le 1 \le 2 \le 2 \le 2 \le 3$

# of k-multisets of an n-element set

$$\mathrm{Mult}\binom{[n]}{k} = \{(a_1, a_2, \dots, a_k) : 1 \le a_1 \le a_2 \le \dots \le a_k \le n\} \xleftrightarrow{F} \{(b_1, \dots, b_k) : 1 \le b_1 < b_2 < \dots < b_k \le n+k-1\}$$

$$(a_1, \dots a_k) \xleftrightarrow{F} (a_1, a_2+1, a_3+2, \dots, a_i+i-1, \dots, a_k+k-1)$$
$$\updownarrow$$
$$\{a_1, a_2+1, \dots \quad a_k+k-1\}$$
$$\cap$$
$$\binom{[n+k-1]}{k}$$

$\boxed{F \text{ is bijection}}$

- $F(a_1, \dots, a_k) \in \binom{[n+k-1]}{k}$
  $1 \le a_1 < a_2+1 < a_3+2 < \dots < a_k+k-1 \le n+k-1$
- F injective $(a_i \ne a_i' \Rightarrow a_i+i-1 \ne a_i'+i-1)$
- F surjective $(1 \le b_1 < b_2 < \dots \le b_k \le n+k-1)$
  $\downarrow$
  $1 \le b_1 \le b_2 - 1 \le b_3 - 2 \le \dots \le b_k - k+1 \le n)$

$$\left|\mathrm{Multi}\binom{[n]}{k}\right| = \left|\binom{[n+k-1]}{k}\right| = \binom{n+k-1}{k}$$

n different balls to k indisunguishable boxes

s.t. NO Box is empty

Def: - partition of a set X is a collection of non-empty subsets of X s.t. each element belongs to exactly 1

$\{X_1, \dots, X_k\}$ s.t. $X_i \neq \emptyset \ \forall i$

$X_1 \cup \dots \cup X_k = X$

- $S(n,k)$ = # of k-partitions of $[n]$

i.e. $\left| \left\{ \{X_1, \dots, X_k\} : \begin{array}{l} X_1 \cup \dots \cup X_k = [n] \\ X_i \neq \emptyset \ \forall i = 1, \dots, k \end{array} \right\} \right|$

Stirling # of the second kind

Remarks:

$S(n,k) = 0 \qquad n < k$

$S(0,0) = 1$ (# of ways to distribute 0 objects into 0 boxes ONE : doing nothing)

Example: $S(n,1) = S(n,n) = 1$ $[n]$

$[n] = \{1\} \cup \{2\} \cup \dots \cup \{n\}$

Example: $S(n, n-1) = \binom{n}{2}$

Meaning: [illegible]

Example: Real Time exercise
Calculate: $S(4,2) = 7$

Thm: $\forall\, k \le n$

$$S(n,k) = S(n-1, k-1) + k \cdot S(n-1, k)$$

Pf: Classify k-partitions of $[n]$ according to placement of n ⇝ singleton $\{n\}$

⇝ $\{\{X_1, \dots, X_k\} \in \mathcal{S}(n,k) : X_i \neq \{n\}\ \forall\, i = 1, \dots, k\}$

$$\downarrow$$

$\{X_1 \setminus \{n\}, \dots, X_k \setminus \{n\}\} \in \mathcal{S}(n-1, k)$

~~#of~~ What if ~~balls~~ boxes ARE distinguishable?

~~Corolly~~

Corollary: $\underbrace{\#\{f:[n]\longrightarrow[k] : f^{-1}(i)\neq\emptyset \ \forall i=1,\dots,k\}}_{\text{surjective fns.}} = k!\,S(n,k)$

Pf: • First create partition of $[n]$ into k nonempty parts

• ~~Order~~ Then Assign parts to elements of $[k]$

Corollary: $\forall x\in\mathbb{C}$ $\forall n\in\mathbb{N}$

$$x^n = \sum_{k=0}^{n} S(n,k)\, x^{\underline{k}}$$

Pf: Both sides are polynomials of degree n

They agree for all $x\in\mathbb{N}$ $\Longrightarrow$ Also agree for $x\in\mathbb{C}$

Left hand side = # fcts from $[n]$ to $[x]$

Classify fns according to size of image $(|f([n])|)$

if $|f([n])| = k \Longrightarrow \exists\ S(n,k)$ k-partitions of $[n]$

$\exists\ x(x-1)\dots(x-k+1)$ ways to assign preimages

$1, x, x^2, \ldots, x^n, \ldots \in \mathbb{C}[x]$ is a basis of $\mathbb{C}[x]$

$1, x, x^{\underline{2}}, x^{\underline{3}}, \ldots, x^{\underline{n}}, \ldots \in \mathbb{C}[x]$ is also a basis

polynoms with coefficient from $\mathbb{C}$

$S_{n,k}$ provides the coefficients to transfer from the second to the first

$$\begin{array}{c} \\ \\ \\ \\ \\ n \rightarrow \\ \\ \end{array} \begin{array}{c} \qquad\qquad k \\ \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & 0 & \\ 1 & 7 & 6 & 1 & & \\ 1 & & & & & \\ \vdots & & S(n,k) & & & \\ \vdots & & & & & \ddots \end{pmatrix} \end{array} \begin{pmatrix} 1 \\ x \\ x^{\underline{2}} \\ x^{\underline{3}} \\ \vdots \\ x^{\underline{n}} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \\ x^n \\ \vdots \end{pmatrix}$$

What is the inverse of the Stirling matrix (of the second kind)

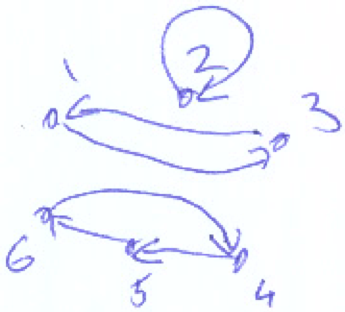
$\rightsquigarrow$ Stirling numbers of the first kind

Cycles in permutation

Recall n-permutation $\pi : [n] \to [n]$ bijective

Example
$$\overset{\pi}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 & 5 & 6 & 4 \end{pmatrix}}$$ or write as word: 321564

DRAW



Group structure $(S_n, \circ)$ $\quad S_n = \{\pi : [n] \to [n] : \pi \text{ bijective}\}$

$\circ$ = composition

Example S_n is non-commutative alredy for $n=3$

Say 312 and 213

Lemma: $\forall\, \pi : [n] \to [n]$ perm $\quad \exists\, i \in [n]$ s.t $\pi^i(x) = x$

$\forall\, x \in [n]$

Pf: $\pi(x), \pi^2(x), \dots, \pi^n(x)$ if x is among them ✓

if x is not among them

$\Rightarrow \pi(x), \dots, \pi^n(x) \in [n] \setminus \{x\}$, n objects in $n-1$ boxes

$\Rightarrow \exists\, y \in [n] \setminus \{x\}$ and $i_1, i_2 \in [n]$

s.t $\pi^{i_1}(x) = y = \pi^{i_2}(x)$

$\Rightarrow \pi^{i_1 - i_2}(x) = x$

$i_1 - i_2 \in [n]$

~~Example: cycles of 321564~~

Def: $\pi \in S_n$, $x \in [n]$

$$i = i(\pi, x) := \min\{ j \in [n] : \pi^{j}(x) = x \}$$

Then $x, \pi(x), \pi^2(x), \ldots, \pi^{i-1}(x)$ form an i-cycle in π

Corollary: All permutations can be decomposed into the disjoint union of cycles.

Example: Cycles of 321564 $\quad (13)(2)(456)$

- (2) is a 1-cycle $\quad$ 2 is a fixed point
- (13) is a 2-cycle $\quad$ (transposition)

Prop # of cyclic permutations $= (n-1)!$

has exactly one cycle $\quad \boxed{(456) = (564) = (645)}$

Pf: $(1\ \_\ \_\ \_\ \ldots\ \_)$

1

What is the image of 1? $(n-1)$ answers

What is the image of the image of 1? $(n-2)$ answers

Def: Stirling number of the first kind

$s_{n,k}$ = # of n-permutations with exactly k-cycles

$$s_{n,0} := \begin{cases} 0 & \text{if } n > 0 \\ 1 & \text{if } n = 1 \end{cases}$$

Examples:

$s_{n,1} = (n-1)!$

$s_{n,n-1} = \binom{n}{2}$

$s_{n,n} = 1$

$s_{n,2} = (n-1)! \cdot H_n$

Real time exercise

1
0 1
0 1 1
0 2 3 1
0 6 11 6 1
0 24 50 35 10 1

Prop: $\displaystyle\sum_{k=0}^{n} s_{n,k} = n!$ $\quad \forall\, n \geq k \geq 0$

Pf: Classify n-permutations according to # of cycles in cycle decomposition

Prop: $s_{n,k} = s_{n-1,k-1} + (n-1)\, s_{n-1,k}$

Pf: Classify according to cycle of n:

Case 1: (n) is a cycle of length 1

$S(n-1,k-1)$ $\qquad \pi \longleftrightarrow \pi$ with (n) deleted is an $(n-1)$-permutation with $k-1$ cycles

Case 2: Map F

$$\pi = (\ldots n x \ldots)(\ldots)\ldots \xrightarrow[\text{Delete } n]{} (\ldots x \ldots)(\ldots)\ldots$$

$\downarrow$

$(n-1)$ perm with k cycles

$(n-1)\, s(n-1,k)$

$\forall$ $(n-1)$-permutation with k cycles occurs $(n-1)$ times as the image under F, because n can be placed in front of ANY of the $n-1$ symbols, each time creating a different n-permutations with k cycles. $\blacksquare$

Prop: $x^{\overline{n}} = \sum_{k=0}^{n} s_{n,k} x^k \quad \forall n \geq 0$

Pf: Induction on n $\qquad n=1 \rightsquigarrow x = s_{1,0} \cdot x^0 + s_{1,1} x$ ✓

$\boxed{n>1}$
$$x^{\overline{n}} = x^{\overline{n-1}} \cdot (x+n-1) = x \cdot x^{\overline{n-1}} + (n-1) x^{\overline{n-1}} = x \cdot \sum_{k=0}^{n-1} s_{n-1,k} x^k + (n-1) \sum_{k=0}^{n-1} s_{n-1,k} x^k$$

$$= \sum_{k=0}^{n-1} s_{n-1,k} x^{k+1} + (n-1) \sum_{k=0}^{n-1} s_{n-1,k} x^k =$$

$$= \sum_{j=1}^{n} s_{n-1,j-1} x^j + \sum_{k=1}^{n} (n-1)\, s_{n-1,k} x^k =$$

$$= \sum_{j=1}^{n} \underbrace{\left(s_{n-1,j-1} + (n-1)\, s_{n-1,k}\right)}_{s_{n,j}} x^j = \sum_{j=0}^{n} s_{n,j} x^j$$

Prop: $$x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} s_{n,k}\, x^{k}$$

Pf:

$$x^{\underline{n}} = (-1)^n (-x)^{\overline{n}} = (-1)^n \sum_{k=0}^{n} s_{n,k} (-x)^k$$

$$= \sum_{k=0}^{n} s_{n,k} (-1)^{n+k} x^k$$

signed Stirling numbers of the first kind

Corollary: $\forall\, i,j$

$$\sum_{k=j}^{i} (-1)^{k-j} S_{i,k}\, s_{k,j} = \delta_{i,j} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

(The inverse of the Stirling matrix of the second kind is the signed Stirling matrix of the first kind.)

The twelvefold ways of counting

How many ways are there to put

n numbered/indistinguishable balls

into r numbered/indistinguishable boxes

such that ~~...~~ $\forall$ box contains at most one ball (placement is injective)

-"- at least -"- (placement is surjective)

or the placement is arbitrary

	injective	surjective	arbitrary
balls numbered, boxes -"-	$r^{\underline{n}}$	$r!\, S_{n,r}$	r^n
balls numbered, boxes indistinguish	0 if $n>r$ 1 if $n \le r$	$S_{n,r}$	$\sum_{q=0}^{r} S_{n,q}$
balls indistinguishable, boxes numbered	$\binom{r}{n}$	$\binom{n-1}{r-1}$	$\binom{n+r-1}{n}$
balls indistinguishable, boxes -"-	0 if $n>r$ 1 if $n \le r$	$p(n,r)$	$\sum_{k=0}^{r} p(n,k)$

$$p(n,r) = \#\left\{(\lambda_1,\dots,\lambda_r) : \begin{matrix}\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_r \ge 1 \\ \lambda_1 + \dots + \lambda_r = n\end{matrix}\right\}$$

"number partitions"

partition of n into r parts

$p(n) = \sum_{r=1}^{\infty} p(n,r) = \sum_{r=1}^{n} p(n,n) = \#$partitions of n into any number of parts

Small values (Real Time Exercise)

$n \backslash k$	0	1	2	3	4			
0	1							
1	0	1						
2	0	1	1					
3	0	1	1	1				
4	0	1	2	1	1			
5	0	1	2	2	1	1		
6	0	1	3	3	2	1	1	
7	0	1	3	4	3	2	1	1

$p(0,0) := 1$

$p(n,0) = 0 \quad \forall n > 0$

- $p(n,1) = 1$
- $p(n,n) = 1$
- $p(n,n-1) = 1$
- $p(n,n-2) = 2$ $\quad 1 \dots 122$, $1 \dots 113$

Solve 6^{th} row $\quad p(6) = 11$

Prop $\quad p(n,k) = p(n-1,k-1) + p(n-k,k)$

Pf: Classify according to whether the smallest part is 1 or NOT

\# of these is $p(n-1,k-1)$

\# of these is $p(n-k,k)$

Bijection

$$\lambda_1 \geq \dots \geq \lambda_k \geq 2 \longleftrightarrow \lambda_1 - 1 \geq \lambda_2 - 1 \geq \dots \geq \lambda_k - 1 \geq 1$$

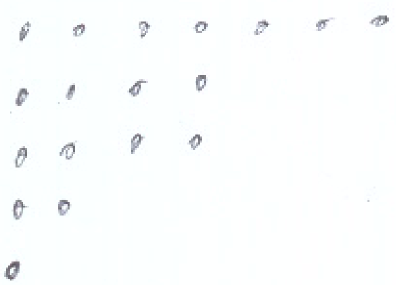
$$\sum_{i=1}^{n} \lambda_i = n \qquad \sum_{i=1}^{k} (\lambda_i - 1) = n - k$$

Prop. ~~8~~ # of partitions of n into at most k parts $(= p(n, \le k))$ $=$ # of partitions of n into parts of <u>size</u> at most k

Pf:

Ferrers diagram

$\lambda = 74421 \rightsquigarrow$



λ_i dots in row i

# of parts = # of rows

Max partsize = # of columns

<u>Conjugate partition λ^*</u> is the one corresponding to the Ferrers diagram we get by reflecting the Ferrers diagram of λ on $y = -x$

(That is: the parts of λ^* are the number of dots in the <u>columns</u>.)

$\lambda^* = 5433111$

~~Prop~~ <u>Def</u>: λ is self-conjugate if $\lambda = \lambda^*$.

<u>Prop</u>: # of partitions of n into distinct odd parts

$=$ # of self-conjugate partitions of n

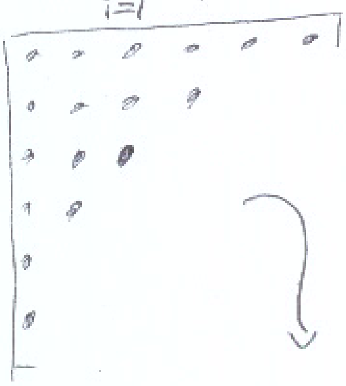
<u>Pf</u>: $\pi_1 \geq \pi_2 \geq \ldots \geq \pi_k$ self-conjugate partition of n

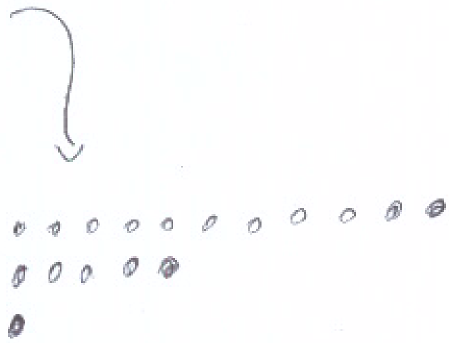
$\sum_{i=1}^{k} \pi_i = n$

$6 \geq 4 \geq 3 \geq 2 \geq 1 \geq 1$

Create partition of n consisting of distinct odd parts

Detach "hook" (first row <u>and</u> column)





$\rightsquigarrow 2\pi_1 - 1 = \sigma_1$ (since π is self conjugate)

<u>Iterate</u> detach first row and column

$$\sigma_2 = 2(\pi_2 - 1) - 1 = 2\pi_2 - 3$$

$$\vdots$$

$$\sigma_j = 2(\pi_j - (j-1)) - 1 = 2\pi_j - (2j-1)$$

σ is a partition of from distinct odd parts

$$\sigma_j = 2\pi_j - (2j-1)$$

$$i > j \quad \underbrace{2\pi_i - (2i-1)}_{\sigma_i} \leq 2\pi_j - (2i-1) < \underbrace{2\pi_j - (2j-1)}_{\sigma_j}$$

- injective
- surjective (from σ one can build back starting at the end)