Leaves, trees, forests...

A graph with no cycle is acyclic. An acyclic graph is called a forest.

A connected acyclic graph is a tree.

Examples. Paths, stars

Theorem (Characterization of trees) For an n-vertex graph G, the following are equivalent

1. G is connected and has no cycles.

2. G is connected and has $n-1$ edges.

3. G has $n-1$ edges and no cycles.

4. For each $u, v \in V(G)$, G has exactly one u, v-path.

Properties of trees

A leaf (or pendant vertex) is a vertex of degree 1.

Lemma. T is a tree, $n(T) \geq 2 \Rightarrow T$ contains at least two leaves.
Deleting a leaf from a tree produces a tree.

A spanning subgraph of G is a subgraph with vertex set $V(G)$.

A spanning tree is a spanning subgraph which is a tree.

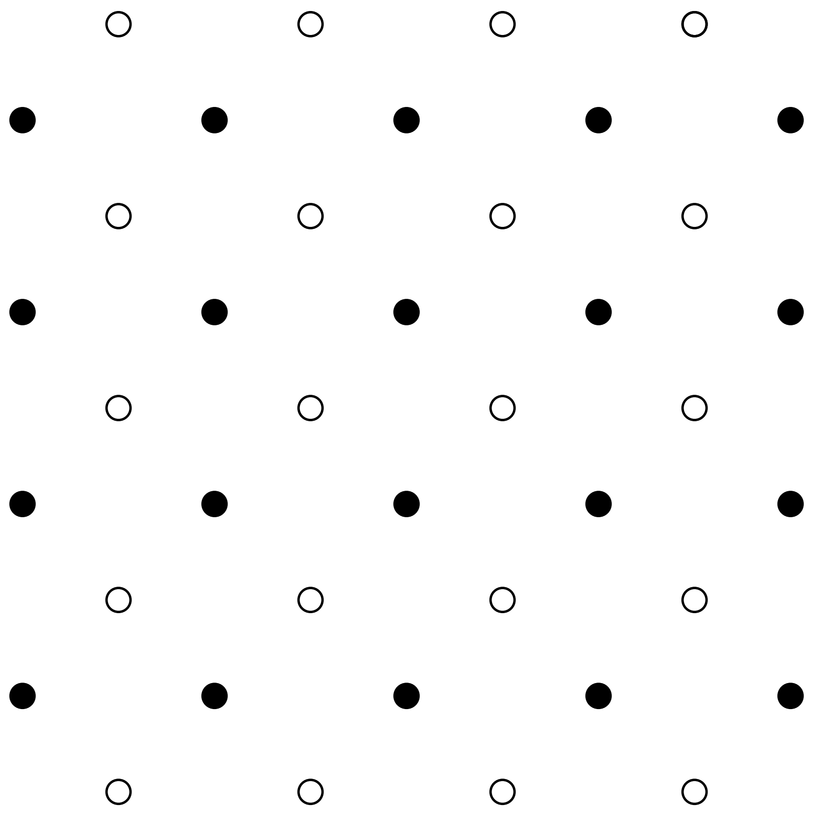
Corollary.

(i) Every edge of a tree is a cut-edge.

(ii) Adding one edge to a tree forms exactly one cycle.

(iii) Every connected graph contains a spanning tree.

Bridg-it* by David Gale



Who wins in Bridg-it?

Theorem. Player 1 has a winning strategy in Bridg-it.

Proof. **Strategy Stealing:**

Suppose Player 2 has a winning strategy.

Then here is a winning strategy for Player 1:

Start with an arbitrary move and then pretend to be Player 2 and play according to Player 2's winning strategy. (Note that playground is symmetric!!) If this strategy calls for the first move of yours, again select an arbitrary edge. Etc...

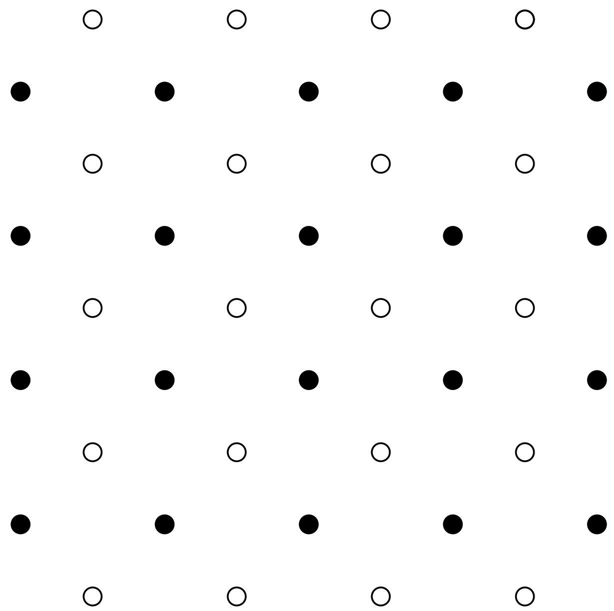
Since you play according to a winning strategy, you win! But we assumed Player 2 also can win $\Rightarrow$ contradiction, since both cannot win.

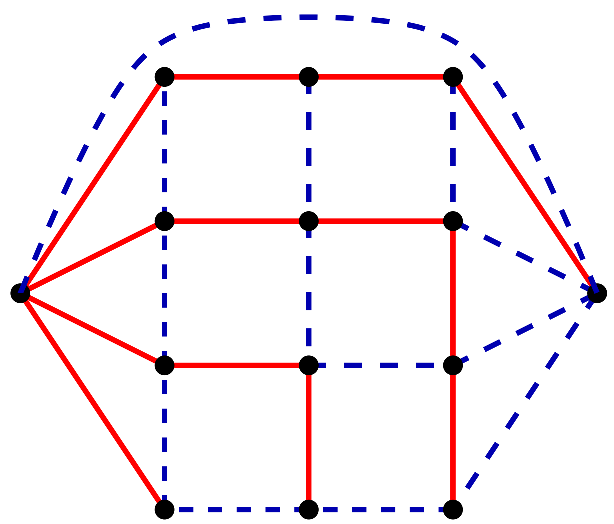
So Player 2 does not have a winning strategy. Also: there is no final position which is a draw. $\square$

Good, but HOW ABOUT AN EXPLICIT STRATEGY???*

*In the *divisor-game* strategy-stealing proves the existence of a sure first player win, but NO explicit strategy is known. Similarly for HEX.

An explicit strategy via spanning trees





The game of “Connectivity”

A positional game is played by two players, Maker and Breaker, who alternately take edges of a base graph G. Maker uses a permanent marker, Breaker uses an eraser. Maker wins the positional game “Connectivity” if by the end he occupies a connected subgraph of G. Otherwise Breaker wins.

Theorem. (Lehman, 1964) Suppose Breaker starts the game. If G contains two edge-disjoint spanning tree, then Maker has an explicit winning strategy in “Connectivity”.

Proof. Maker maintains two spanning trees T_1 and T_2, such that after each full round,

(i) $E(T_1) \cap E(T_2)$ consists of the edges claimed by Maker,

(ii) $E(T_1) \triangle E(T_2)$ contains only unclaimed edges.

Remark. The other direction of the Theorem is also true.

The tool for Player 1. (i.e. Maker)

Proposition. If T and T' are spanning trees of a connected graph G and $e \in E(T) \setminus E(T')$, then **there is** an edge $e' \in E(T') \setminus E(T)$, such that $T - e + e'$ is a spanning tree of G.

Proposition. If T and T' are spanning trees of a connected graph G and $e \in E(T) \setminus E(T')$, then **there is** an edge $e' \in E(T') \setminus E(T)$, such that $T' + e - e'$ is a spanning tree of G.

Counting labeled trees

How many trees are there on vertex set $[n]$?

Example: $n = 1, 2, 3, 4, 5$... Conjecture?

Theorem The number of trees on $[n]$ is n^{n-2}.

Proof. (Prüfer code)

Bijection p from family of n-vertex trees to $[n]^{n-2}$.
Define $p(T) \in [n]^{n-2}$:

Let $T_0 = T$. Iteratively for $i = 1, \ldots n-2$ **do**

(1) define $p(T)_i$ to be the (unique) neighbor of the smallest leaf ℓ_i of T_{i-1}

(2) delete ℓ_i to obtain $T_i := T_{i-1} - \ell_i$

This is a bijection!

Inverse: Given vector $(p_1, \ldots, p_{n-2}) \in [n]^{n-2}$, for $1 \le i \le n-1$, iteratively define:

$b_i := \min\left([n] \setminus \{b_1, \ldots, b_{i-1}, p_i, \ldots, p_{n-2}\}\right)$ and p_{n-1} by $[n] \setminus \{b_1, \ldots, b_{n-1}\} := \{p_{n-1}\}$.

Note: (1) $b_i \neq b_j$ for $i \neq j$ (2) $b_i \neq p_j$ for $j \geq i$

Define G_i by $V(G_i) := \{b_i, \ldots, b_{n-1}, p_{n-1}\}$

$$E(G_i) := \{p_j b_j : j = i, \ldots, n-1\}$$

G_i is well-defined: $p_i \in V(G_{i+1}) \subseteq V(G_i)$ since by (1) and (2): $V(G_{i+1}) = [n] \setminus \{b_1, \ldots, b_i\}$ and by (2) p_i is different from the vertices $b_1, \ldots, b_i$

Claim G_i is a tree and $[n]\setminus\{b_1, \ldots, b_{i-1}, p_i, \ldots, p_{n-2}\}$ is the set of its leaves. In particular b_i is the smallest leaf of G_i, $G_{i+1} = G_i - b_i$ and $p(G_i) = (b_i, \ldots, b_{n-2})$.

Proof: Induction for $i = n-1, n-2, \ldots, 1$.

$V(G_{n-1}) := \{b_{n-1}, p_{n-1}\}$ and $G_{n-1} \cong K_2$.

changes to the leaves from G_{i+1} to G_i:

New leaf: by (1) $b_i \notin V(G_{i+1})$, so its only neighbor in G_i is p_i.

p_i already has a neighbor in G_{i+1}, so it is not a leaf in G_i