Connectivity

A vertex cut of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component.

For $G \neq K_n$, the connectivity of G is

$$\kappa(G) := min\{|S| : S \text{ is a vertex cut}\}.$$

By definition, $\kappa(K_n) := n - 1$.

A graph G is k-connected if (1) $v(G) \geq k + 1$ and (2) there is no vertex cut of size $k - 1$. (i.e. $\kappa(G) \geq k$)

Examples.
$$\begin{aligned} \kappa(K_{n,m}) &= \text{min}\{n, m\} \\ \kappa(Q_d) &= d \end{aligned}$$

Edge-connectivity

An edge cut of a multigraph G is an edge-set of the form $[S, \bar{S}]$, with $\emptyset \neq S \neq V(G)$ and $\bar{S} = V(G) \setminus S$.

For $S, T \subseteq V(G)$, $[S, T] := \{xy \in E(G) : x \in S, y \in T\}$.

The edge-connectivity of G is

$$\kappa'(G) := \min\{\ |[S, \bar{S}]|\ :\ [S, \bar{S}] \text{ is an edge cut}\}.$$

A graph G is k-edge-connected if there is no edge cut of size $k - 1$ (i.e. $\kappa'(G) \geq k$).

Theorem. (Whitney, 1932) If G is a simple graph, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Homework. Example of a graph G with $\kappa(G) = k$, $\kappa'(G) = l$, $\delta(G) = m$, for any $0 < k \leq l \leq m$.

Characterization of 2-connected graphs

Theorem. Let G be a graph with $v(G) \geq 3$. Then the following four statements are equivalent.

(i) G is 2-connected

(ii) for every $u, v \in V(G)$ there exist two internally disjoint u, v-paths in G. (Whitney, 1932)

(iii) For all $x, y \in V(G)$, there is a cycle through x and y.

(iv) $\delta(G) \geq 1$, and every pair of edges of G lies on a common cycle.

Proof. $(i) \Rightarrow (ii)$ Induction on the distance between u and v. Let w be the penultimate vertex on a shortest path from u to v. Combine the edge vw and a u, v-path P of $G - w$ with two internally disjoint u, w-paths R and Q of G to find two internally disjoint u, v-paths.

For $(iii) \Rightarrow (iv)$:

Expansion Lemma. Let G' be a supergraph of a k-connected graph G obtained by adding one vertex to $V(G)$ with at least k neighbors. Then G' is k-connected as well.

Menger's Theorem

Given $x, y \in V(G)$, a set $S \subseteq V(G) \setminus \{x, y\}$ is an x, y-separator (or an x, y-cut) if $G - S$ has no x, y-path.
A set $\mathcal{P}$ of paths is called pairwise internally disjoint (p.i.d.) if for any two path $P_1, P_2 \in \mathcal{P}$, P_1 and P_2 have no common internal vertices.
Define

$$\kappa(x, y) := \min\{|S| : S \text{ is an } x, y\text{-cut,}\} \text{ and}$$
$$\lambda(x, y) := \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.i.d. } x, y\text{-paths}\}$$

Local Vertex-Menger Theorem (Menger, 1927) Let $x, y \in V(G)$, such that $xy \notin E(G)$. Then

$$\kappa(x, y) = \lambda(x, y).$$

Corollary (Global Vertex-Menger Theorem) A graph G is k-connected iff for any two vertices $x, y \in V(G)$ there exist k p.i.d. x, y-paths.

Proof: **Lemma.** For every $e \in E(G)$, $\kappa(G - e) \geq \kappa(G) - 1$.

Edge-Menger

Given $x, y \in V(G)$, a set $F \subseteq E(G)$ is an x, y-disconnecting set if $G - F$ has no x, y-path. Define

$$\kappa'(x, y) := \min\{|F| : F \text{ is an } x, y\text{-disconnecting set,}\}$$
$$\lambda'(x, y) := \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.e.d.}^* \; x, y\text{-paths}\}$$

* p.e.d. means pairwise edge-disjoint

Local Edge-Menger Theorem For all $x, y \in V(G)$,

$$\kappa'(x, y) = \lambda'(x, y).$$

Corollary (Global Edge-Menger Theorem) Multigraph G is k-edge-connected iff there is a set of k p.e.d.x, y-paths for any two vertices x and y.