1 Identities with binomial coefficients

We will present some properties of the Pascal Triangle.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
\iddots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

Entry in row n and diagonal i is $\binom{n}{i}$. Counting rows and diagonals from 0.
Last week: Sum of entries in one row is

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

and the Pascal Triangle is symmetric:

$$\binom{n}{i} = \binom{n}{n-i}$$

Pascal himself used in his *Traité du triangle arithmétique* (1654) the first ever printed induction proof to prove the following observation:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
\iddots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

The ratio of two neighbouring entries is equal to the ratio of their distances to the upper left or upper right rim of the triangle:

$$\frac{15}{20} = \frac{3}{4}.$$

$\rightsquigarrow$ exercise for fun.
Instead we prove that

$$\binom{n-1}{k} + \binom{n-1}{k+1} = \binom{n}{k+1}$$

holds for $0 \leq k < n$ and $k, n \in \mathbb{N}$ with a combinatorial argument:

Proof. Let

$$f : \begin{cases} \binom{[n-1]}{k} \cup \binom{[n-1]}{k+1} & \to \binom{[n]}{k+1} \\ A & \mapsto \begin{matrix} A \cup \{n\}, & \text{if } |A| = k \\ A, & \text{if } |A| = k+1 \end{matrix} \end{cases}$$

f is surjective (we hit every element of $2^{[n]}$) and injective (unique preimages), so f is bijective. So bijection rule gives the statement. □

Remember now the following identity:

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

Derived a week ago by combinatorial arguments with $0 \leq k \leq n$ and k, n integers.
But: why stick to these integer values when dealing with binomial coefficients?
In fact, we could (and will) consider

$$z^{\underline{k}} = z(z-1)\cdots(z-k+1)$$

and

$$z^{\overline{k}} = z(z+1)\cdots(z+k-1)$$

from now on as polynomials in over $\mathbb{C}$ (and still call them the *falling* or *rising factorials*, resp.).
Observation: Let $k \in \mathbb{N}$. Then $z^{\underline{k}}$ has zeroes at $z = 0, 1, 2, \ldots, k-1$ and $z^{\overline{k}}$ has zeroes at $z = 0, -1, -2, \ldots, -k+1$.

Next, we define:

$$n^{\underline{0}} := 1, n^{\overline{0}} := 1, \text{ and } 0! := 1.$$

What about k in $\binom{n}{k}$? $k < 0$ would make it necessary to continue the factorial to negative values ($\rightsquigarrow$ number theory). We do it easier and define for any $z \in \mathbb{C}$ and $k \in \mathbb{Z}$:

$$\binom{z}{k} := \begin{cases} \frac{z(z-1)(z-2)\cdots(z-k+1)}{k!} & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases}$$

Observation from above implies that $\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = 0$ for integral $k > n$.

Theorem 1. *The recurrence is still valid for $z \in \mathbb{C}$ and $k \geq 0$.*

$$\binom{z-1}{k} + \binom{z-1}{k+1} = \binom{z}{k+1}$$

Proof. Both sides are polynomials in $\mathbb{C}$ that coincide for all integer values of z. But two polynomials with degree ℓ that coincide in $\ell+1$ points are identical ($\rightsquigarrow$ algebra/analysis). $\square$

This *polynomial trick* will be of great use later! (What about $k < 0$?)
Now we will return to using our "new" binomials with integers.

Theorem 2 (Binomial Theorem)**.** *For all integral $n \geq 0$,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof. Induction:
1) True for $n = 0$, since

$$(x+y)^0 = 1 = \binom{0}{0} x^0 y^0$$

2) Now we assume that the statement is true for all $k \leq n$ $(*)$. Hypothesis: It also holds for $n+1$.

$$(x+y)^{n+1} = (x+y)^n(x+y) =$$

$$x(x+y)^n + y(x+y)^n \stackrel{*}{=} x\sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i} + y\sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i} =$$

$$\sum_{i=-1}^{n}\binom{n}{i}x^{i+1}y^{n-i} + \sum_{i=0}^{n+1}\binom{n}{i}x^i y^{n-i+1} = \sum_{i=-1}^{n}\binom{n}{i}x^{i+1}y^{n-i} + \sum_{i=-1}^{n}\binom{n}{i+1}x^{i+1}y^{n-i} =$$

$$\sum_{i=-1}^{n}\binom{n+1}{i+1}x^{i+1}y^{n-i} = \sum_{i=0}^{n+1}\binom{n+1}{i}x^i y^{n-i+1}$$

$\square$

Applications of this theorem:

(i) Let $x = 1$ and $y = 1$. Then we get

$$(1+1)^n = \sum_{i=0}^{n}\binom{n}{i}1^i 1^{n-i}.$$

So:

$$2^n = \sum_{i=0}^{n}\binom{n}{i}$$

(ii) Let $x = -1$ and $y = 1$ **and** $n \neq 0$. Then we get

$$(-1+1)^n = 0^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i = -\sum_{\substack{0 \le i \le n \\ i \text{ odd}}} \binom{n}{i} + \sum_{\substack{0 \le i \le n \\ i \text{ even}}} \binom{n}{i}.$$

So:

$$\sum_{\substack{0 \le i \le n \\ i \text{ odd}}} \binom{n}{i} = \sum_{\substack{0 \le i \le n \\ i \text{ even}}} \binom{n}{i}$$

Because of $n \neq 0$, any **non-empty** set has the same number of subsets with even cardinality and subsets with odd cardinality. (Empty set has one even subset $\varnothing$ and no odd subset.)

Alternative: a combinatorial proof for ii). Let $[n]$ be the set. We define

$$f : \begin{cases} 2^{[n]} & \to 2^{[n]} \\ A & \mapsto \begin{matrix} A \cup \{n\} & \text{if } n \notin A \\ A \setminus \{n\} & \text{if } n \in A \end{matrix} \end{cases}$$

- Again: surjective (we hit every element of $2^{[n]}$) and injective (unique preimages), so it is bijective.
- In particular it maps all odd subsets to even subsets and vice versa.
- As it is also bijective if restricted to $\{A \subseteq 2^{[n]} : |A| \text{ is odd}\}$, we get by bijection rule the statement.

Other identities:

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

for $1 \le k \le n$.

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
\iddots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

$$
\begin{aligned}
\sum_{i=k}^{n}\binom{i}{k} = \binom{k+1}{k+1} + \sum_{i=k+1}^{n}\binom{i}{k} = \\
&\binom{k+2}{k+1} + \sum_{i=k+2}^{n}\binom{i}{k} = \binom{k+2}{k+1} + \binom{k+2}{k} + \sum_{i=k+3}^{n}\binom{i}{k} = \\
&\binom{k+3}{k+1} + \sum_{i=k+3}^{n}\binom{i}{k} = \ldots = \binom{n+1}{k+1}
\end{aligned}
$$

Theorem 3 (Vandermonde-Identity)**.** *For $n \geq 0$ and $x, y \in \mathbb{C}$ holds that*

$$\sum_{i=0}^{n}\binom{x}{i}\binom{y}{n-i} = \binom{x+y}{n}$$

Proof. With polynomial trick: We prove at first the statement combinatorially for $x, y \in \mathbb{N}$, then we deal with $x, y \in \mathbb{C}$.
Let X and Y be two sets with $X \cap Y = \varnothing$ and $|X| = x$ and $|Y| = y$.
On the right side, we have the number of n-subsets of $X \cup Y$.
Any of these subsets has i elements from X and $n-i$ elements from Y, where $0 \leq i \leq n$. We classify the subsets by i.
For each i, we have $\binom{y}{i}\binom{y}{n-i}$ possibilities for that (product rule). The sum rule gives the result.
Now: $x, y \in \mathbb{C}$. So both sides are polynomials in two variables that coincide for all integers. So both sides coincide also for $x, y \in \mathbb{C}$. $\square$

Consequences:

$$\frac{(x+y)^{\underline{n}}}{n!} = \sum_{i=0}^{n}\frac{x^{\underline{i}}y^{\underline{n-i}}}{i!(n-i)!} \Rightarrow (x+y)^{\underline{n}} = \sum_{i=0}^{k}\binom{n}{i}x^{\underline{i}}y^{\underline{n-i}}$$

and similarly

$$(x+y)^{\overline{n}} = \sum_{i=0}^{k}\binom{n}{i}x^{\overline{i}}y^{\overline{n-i}}.$$

2 Multisets

Set: All elements are different. Multiset: This is not necessary anymore.
Example: $M = \{1, 1, 2, 3\}$ is a multiset over the set $S = \{1, 2, 3\}$, element 1 comes with multiplicity 2.

Cardinality of a multiset: number of elements counted with their multiplicity, so $|M| = 4$.
Last week: (k-)words as ordered k-sets. Similary one can define (k-)words as ordered k-multisets (i.e. words where letters can appear more than once).
Example: 123321 is a 6-word where each letter appears twice.

Theorem 4. *Number of n-words over alphabet (ground set) $\{a_1, a_2, \ldots, a_k\}$, where letter a_i appears m_i times, is*

$$\frac{n!}{m_1!m_2!\cdots m_k!}.$$

Proof. For the first class of letters, we choose an m_1-subset of $[n]$. For second class, we choose m_2-subset of $[n-m_1]$ and so on. So number of possibilities:

$$\binom{n}{m_1}\binom{n-m_1}{m_2}\binom{n-m_1-m_2}{m_3}\cdots\binom{m_k}{m_k} = \frac{n^{\underline{m_1}}(n-m_1)^{\underline{m_2}}\cdots m_k^{\underline{m_k}}}{m_1!m_2!\cdots m_k!} = \frac{n!}{m_1!m_2!\cdots m_k!}$$

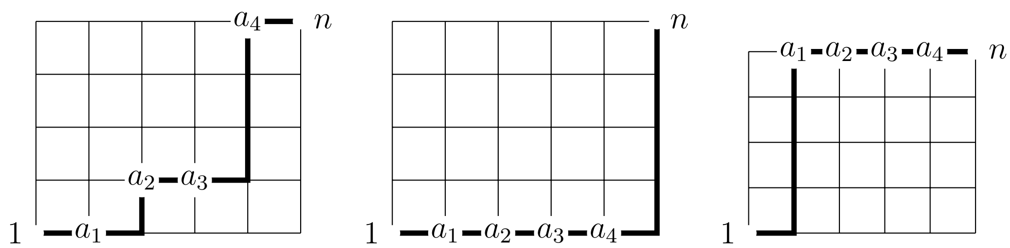
$\rightsquigarrow$ exercise? $\square$

Theorem 5. *Number of k-multisets over an n-set:*

$$\frac{n(n+1)\cdots(n+k-1)}{k!} = \frac{n^{\overline{k}}}{k!}$$

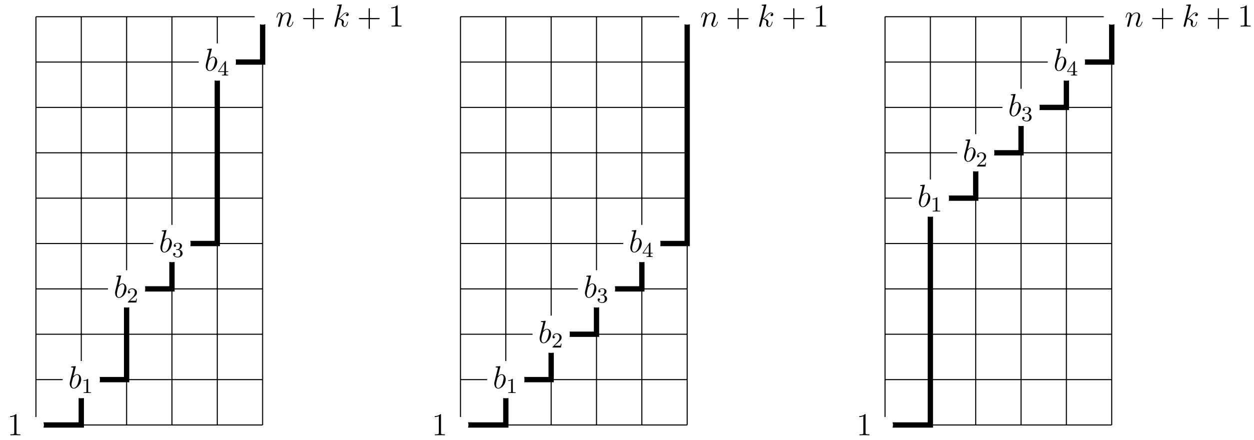
Proof. For the proof we will interpret a k-multiset over $[n]$ as a non-decreasing sequence of length k of integers between 1 and n:

$$1 \leq a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_k \leq n$$



Compare with sequences of *strictly increasing* integers between 1 and $n+k-1$:

$$1 < b_1 < b_2 < b_3 < \ldots < b_k < n+k+1$$



So we are relating the following two sets:

$$A := \{(a_1, a_2, \dots, a_k) : 1 \le a_1 \le a_2 \le \dots \le a_k \le n\}$$

and

$$B := \{(b_1, b_2, \dots, b_k) : 1 < b_1 < b_2 < \dots < b_k < n+k+1\}.$$

We can map A to B with the following map:

$$f : \begin{cases} A & \to B \\ (a_1, a_2, \dots, a_k) & \mapsto (b_1, b_2, \dots, b_k) \\ \text{such that} & b_i = i + a_i \quad \forall i \in [k] \end{cases}$$

This map f is bijective:

- It is surjective (for every lattice path induced by an element of B there is a lattice path induced by element of A)
- It is injective (unique preimages)

Size of B: $\binom{n+k-1}{k}$ possibilities to choose positions of b_i, that is:

$$\binom{n+k-1}{k} = \frac{n^{\overline{k}}}{k!}$$

possibilities. □

Notation:

$$\left(\!\!\binom{n}{k}\!\!\right) := \binom{n+k-1}{k} = \frac{n^{\overline{k}}}{k!}$$