Discrete Mathematics 1 — Summer Term 2015

email: andreas.loos@math.fu-berlin.de — 30 June/2 July 2015

Mock Exam – Solutions

Show all your work and state precisely the theorems you are using from the lecture. Ideally, try to solve the sheet within a time limit of 90 minutes, without using any books, notes, etc ... (but of course this is not mandatory if you feel it would not yet make sense this way). It will be graded like the Final Exam, but the points do **not** count towards your exercise credit.

Problem 1 [10 points]

Prove the following statement for each graph G: Either G or its complement $\overline{G}$ is connected.

Solution:

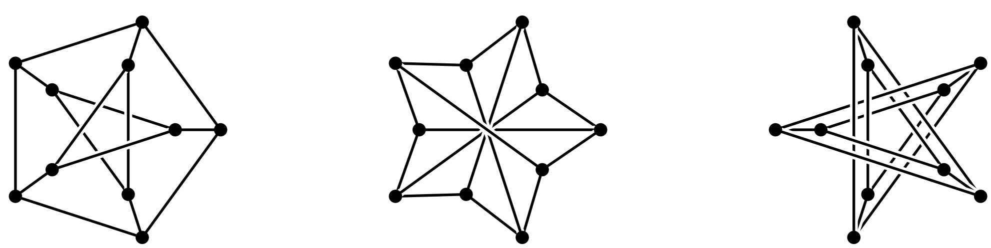
Assume G is disconnected. We show that then $\overline{G}$ is connected, that is for any pair of vertices $x, y \in V(G)$ there is an x, y-path in $\overline{G}$.

Let $A \subseteq V(G)$ be a connected component and let $B = V(G) \setminus A$ be its complement. Then for each pair of nodes $a \in A$ and $b \in B$ we have $\{a, b\} \notin E(G)$, that is $\{a, b\} \in E(\overline{G})$. So there is an a, b-path of length one in $\overline{G}$.

Moreover, for each pair of nodes $a, a' \in A$ it holds that a and a' are connected in $\overline{G}$ via path of length two. Indeed, for any $b \in B$, we have $\{a, b\} \notin E(G)$ and $\{a', b\} \notin E(G)$, so a, b, a' is a path of length two in $\overline{G}$. Similarly, for any $b, b' \in B$ and $a \in A$, the path b, a, b' is contained in $\overline{G}$.

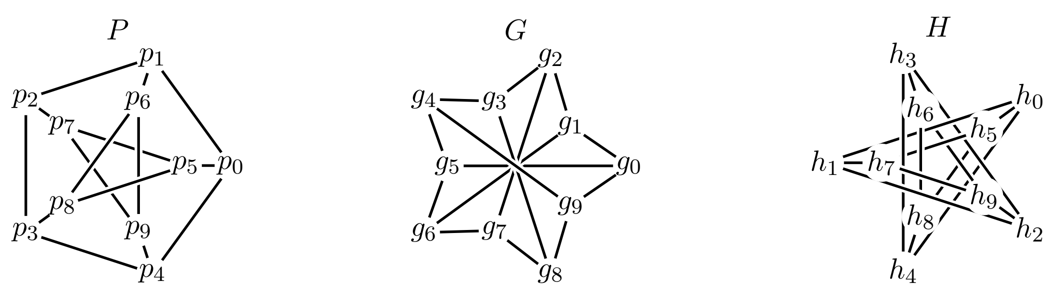
Problem 2 [10 points]

Prove which of the following graphs are isomorphic to each other and which are not.



Solution:

First, we add some notation:



Any two of the three graphs are not isomorphic to each other. The arguments can go as follows:

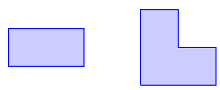
P and G: Graph P is the Petersen graph which is not bipartite, since it contains odd cycles (for instance induced by node set $\{p_0, p_1, p_2, p_3, p_4\}$). In contrast, graph G is bipartite: For every i, j with $i = j \bmod 2$, we have $\{g_i, g_j\} \notin E(G)$. So the vertex set can be partitioned into two independent sets: the vertices with an odd index and the vertices with an even index set. Hence P cannot be isomorphic to G.

G and H: G is bipartite, H contains a 5-cycle $\{h_0, h_1, h_2, h_3, h_4\}$. Hence G cannot be isomorphic to H.

P and H: There is no 4-cycle in P (as it was proved in the lecture). In H, however, there is a 4-cycle: $\{h_0, h_1, h_7, h_5\}$. So P and H are also not isomorphic.

Problem 3 [10 points]

Write down a recurrence relation for the sequence c_n, where c_n represents the number of ways one can cover a $n \times 2$-rectangle completely without overlap with the following two types of tiles:



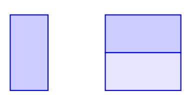
The tiles may be rotated by integral multiples of 90 degree. It is not necessary to *solve* the recurrence.

Solution:

We classify the possibilities by what which tile is put in the left upper corner.

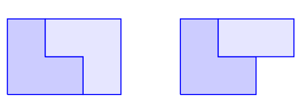
One possibility is to put the 2×1-rectangle upright. In this case there are c_{n-1} possibilities to finish the tiling.

Another possibility is to put it horizontally which forces us to use another copy of this rectangle below. In this case there are c_{n-2} possibilities to complete the tiling.



Moreover, there are 2 further possibilities, using the L-shaped tiles:

In both cases, we can either proceed by completing a rectangle with a turned version of the L-tile or by using a rectangle 1×2. The latter option produces another „open“ form, the length of which is one bigger. This case splits again to two subcases depending on whether we insert an L-tile and close it or lengthen it with a rectangle 1×2 and leave it open, etc ...



$\dots$

Hence, for each $k \geq 3$ we have two possibilities to construct a $2 \times k$-rectangle starting and ending with an L-tile and containing only 1×2 rectangles in between.
So $c_0 = 1, c_1 = 1, c_2 = 2$ and for $n \geq 3$

$$c_n = c_{n-1} + c_{n-2} + 2\sum_{k=3}^{n} c_{n-k}$$

Remark. Note that we can get a recurrence if we subtract c_{n-1} from c_n (which cancels out most of the sum):

$$c_n - c_{n-1} = c_{n-1} + c_{n-2} + 2c_{n-3} - c_{n-2} - c_{n-3}$$

So:

$$c_n = 2c_{n-1} + c_{n-3}.$$

Problem 4 [10 points]

(a) Define the Ramsey number $R(k, l)$.
(b) Prove that $R(k, k) > \sqrt{2}^k$ for every large enough k. (The statement is true for every k, but here it is enough if you show it for, say, $k \geq 10$).

Solution:

(a) The Ramsey number $R(k, l)$ is the smallest integer n such that for *every* edge-coloring $c : E(K_n) \to \{\texttt{red}, \texttt{blue}\}$ of the complete graph K_n there is either a monochromatic red clique (that is: a complete subgraph with only red edges) of order k or a blue clique of order l.
(b) We need to show that there exists a two-coloring of the edges of $K_{\lfloor \sqrt{2}^k \rfloor}$ which does not contain a monochromatic clique of order k.
The number of two-colorings of the edges of K_n is

$$A := 2^{\binom{n}{2}}.$$

We next consider the number B of all colorings such that K_n contains at least one monochromatic k-clique.
The strategy will be as follows: If $A - B > 0$, then there must be a coloring which *does not* contain a monochromatic k-clique. So the Ramsey number $R(k)$ is larger than any n that allows $A > B$ to happen.
The number of colorings such that a fixed k-subset $S \subseteq V(K_n)$ becomes a monochromatic k-clique is:

$$2 \cdot 2^{\binom{n}{2} - \binom{k}{2}}$$

(The color of the $\binom{k}{2}$ edges in $\binom{S}{2}$ is either all $\texttt{red}$ or all $\texttt{blue}$, while the remaining $\binom{n}{2} - \binom{k}{2}$ edges can each be $\texttt{red}$ or $\texttt{blue}$)
Summing up over all k-cliques S, we need that

$$B \leq \binom{n}{k} \cdot 2 \cdot 2^{\binom{n}{2} - \binom{k}{2}} \overset{!}{<} 2^{\binom{n}{2}} = A$$

Since $\binom{n}{k} \le \left(\frac{ne}{k}\right)^k$ (by lecture), we are done if n and k satisfies

$$\left(\frac{ne}{k}\right)^k > 2^{\frac{k-1}{2}k-1}.$$

That is if

$$n > 2^{\frac{k}{2}}\sqrt{2}\cdot k \cdot \frac{1}{e}\frac{1}{2^{\frac{1}{k}}} > 2^{\frac{k}{2}}$$

since $\sqrt{2}\cdot k\cdot \frac{1}{e}\frac{1}{2^{\frac{1}{k}}} \approx 0.5 \cdot k \cdot \underbrace{\frac{1}{2^{\frac{1}{k}}}}_{\to 1 \text{ for } k\to\infty} > 1$ for $k \ge 3$.

Problem 5 [10 points]

(a) Define the Stirling numbers $S_{n,k}$ of the second kind.

(b) Prove that it holds that

$$S_{n,3} = \frac{3^n - 3\cdot 2^n + 3}{6}$$

Solution:

(a) The Stirling numbers of the second kind is the number of ways to partition an n-element set into k *nonempty* and *pairwise disjoint* subsets.

(b) We will use induction on n. For the base case:

$$S_{3,3} = 1 = \frac{3^3 - 3\cdot 2^3 + 3}{6}$$

For the general case we use the following formulas:
- $S_{n,2} = \frac{1}{2}(2^n - 2)$ and that
- $S_{n,k} = kS_{n-1,k} + S_{n-1,k-1}$

The second equation was proved in the lecture. (Alternatively, a short proof can be given by classifying based on the partition class of the element n. If $\{n\}$ is a partition class on its own, then we have a partition of the remaining $(n-1)$ elements into $k-1$ non-empty sets. Otherwise n is not alone in a set and we can insert it into any of the k nonempty sets of any k-partition of $[n-1]$.)

For the first formula: we have $2^n - 2$ possibilities to put n items into Box 1 or Box 2, such that none of the boxes are empty. Since their order does not matter for a 2-partition, we have to divide by two. Let $n > 3$. Then using the formulas and then induction we have

$$\begin{aligned} S_{n,3} &= 3S_{n-1,3} + S_{n-1,2} = 3\frac{3^{n-1} - 3\cdot 2^{n-1} + 3}{6} + 2^{n-2} - 1 \\ &= \frac{3^n - 3^2\cdot 2^{n-1} + 3^2 + 3\cdot 2^{n-1} - 6}{6} = \frac{3^n - 3\cdot 2^n + 3}{6} \end{aligned}$$

Alternative solution: We can argue that $S_{n,3}$ is the number of surjections from $[n]$ to $[3]$, divided by $3!$. Indeed, a surjection can be identified by an ordered triple of nonempty subsets (A_1, A_2, A_3) that partition $[n]$. For a 3-partition the order of these three subsets does not matter, hence:

$$S_{n,3} = \frac{\left|\left\{f : [n] \to [3] : f([n]) = [3]\right\}\right|}{3!}$$

The number of surjections from $[n]$ to $[3]$ can be computed by inclusion-exclusion:

$$\underbrace{3^n}_{\text{number of all functions}} - \underbrace{3 \cdot 2^n}_{\text{at least one image is } \textit{missing}} + \underbrace{3 \cdot 1^n}_{\text{two images are } \textit{missing}}.$$

Problem 6 [10 points]

For $n \in \mathbb{N}$, let i_n denote the number of permutations $f \in S_n$ having the property $f(f(x)) = x$ for all $x \in [n]$. Define $i_0 := 1$. Prove the recurrence

$$i_n = i_{n-1} + (n-1)i_{n-2}$$

and find the exponential generating function of the sequence.

Solution:

Part one: These are the involutions. Each such permutation has only 1- and 2-cycles. Hence we can classify them according to the last element n. Either it is in a 1-cycle or not. If it is, we have i_{n-1} possibilites to group the remaining $n-1$ elements into 1- and 2-cycles. If it is not, we have a pair of n and one of the $n-1$ other elements ($\rightsquigarrow n-1$ possibilites). The remaining $n-2$ elements give i_{n-2} possibilites.

Part two: We define $i_{-1} := 0$. (In fact, the actual value doesn't interest, and this is not even necessary by starting with $i = 2$ for instance.) Write for the generating function $\hat{I}(x)$. We multiply the recurrence with $\frac{x^{n-1}}{(n-1)!}$ and sum up from $n = 1$ to ∞. Hence we get

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} i_n = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} i_{n-1} + \sum_{n=1}^{\infty} (n-1) \frac{x^{n-1}}{(n-1)!} i_{n-2} \Leftrightarrow$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} i_{n+1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} i_n + x \sum_{n=0}^{\infty} \frac{x^{n-1}}{(n-1)!} i_{n-1} \overset{i_{-1}=0}{\Leftrightarrow}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} i_{n+1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} i_n + x \sum_{n=0}^{\infty} \frac{x^n}{n!} i_n$$

$$\Rightarrow \frac{d}{dx} \hat{I}(x) = \hat{I}(x) + x\hat{I}(x) = (1+x)\hat{I}(x)$$

Hence

$$\int \frac{d\hat{I}}{\hat{I}} = \int (1+x)dx$$

or

$$\log(\hat{I}) = x + \frac{1}{2}x^2 + C$$

or

$$\hat{I}(x) = e^{x + \frac{1}{2}x^2 + C}$$

Since $I(0) = 1$, we get that $C = 0$.