Posets

$(P, \leq)$ is a poset (*partially ordered set*) if the relation $\leq$ on P is

- reflexive ($a \leq a$ for all $a \in P$)
- antisymmetric ($a \leq b$ and $b \leq a \Rightarrow a = b$)
- transitive ($a \leq b$ and $b \leq c \Rightarrow a \leq c$)

Definition a and b are comparable if $a \leq b$ or $b \leq a$. Otherwise a and b are incomparable.

Representation: Hasse diagram

Examples:

• $\mathbb{R}$ (or $\mathbb{Q}$ or $\mathbb{Z}$ or $\mathbb{N}$) with $\leq$ (usual order) is a poset
(No two incomparable elements: total order)

• S is a set, then $(2^S, \subseteq)$ is a poset; Boolean poset

• n is an integer, then $\{x \in [n] : x|n\}$ with the divisibility relation $|$ is a poset

$C \subseteq P$ is a chain if any two elements are comparable.

$A \subseteq P$ is an antichain if no two elements are comparable.

Largest antichains

The width of a poset is the size of the largest antichain.

Sperner's Theorem The width of the Boolean poset is $\binom{n}{\lfloor n/2 \rfloor}$.

Reformulation: How many subsets of $[n]$ can be select if it is forbidden to select two sets such that one is subset of the other?

You can select all $\binom{n}{k}$ subsets of a given size k: they certainly satisfy the property.
$k = \left\lfloor \frac{n}{2} \right\rfloor$ maximizes their number.

Sperner's Theorem If $\mathcal{F} \subseteq 2^{[n]}$ is a family of subsets such that for every $A, B \in \mathcal{F}$ we have $A \not\subseteq B$ then

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Permutation method

Proof. Count permutations $\pi \in S_n$ of $[n]$ which have an initial segment from $\mathcal{F}$. Formally, double-count

$$M = |\{(\pi, F) : \pi \in S_n, F \in \mathcal{F}, F = \{\pi(1), \ldots, \pi(|F|)\}\}|$$

For every $F \in \mathcal{F}$ there are $|F|!(n-|F|)!$ permutations $\pi \in S_n$ with $\{\pi(1), \ldots, \pi(|F|)\} = F$. So

$$M = \sum_{F \in \mathcal{F}} |F|!(n-|F|)!.$$

For every $\pi \in S_n$ there is at most one k such that $\{\pi(1), \ldots, \pi(k)\} \in \mathcal{F}$.

So $M \le n!$.

Hence

$$\sum_{F \in \mathcal{F}} |F|!(n-|F|)! \ \le \ n!$$

$$1 \ge \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \ge \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = |\mathcal{F}| \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

Min-max statement for max-chains

A partition $\mathcal{C} = \{C_1, \ldots, C_l\}$ of P is a chain partition of P if all C_is are chains.

A partition $\mathcal{A} = \{A_1, \ldots A_k\}$ is an antichain partition of P if all A_is are antichains.

Proposition $\max\{|C| : C \text{ is a chain}\} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is an antichain partition of } P\}$

Proof. $\boxed{\leq}$ is immediate.

$\boxed{\geq}$ The set $A = \{x \in P : x \not\leq y \text{ for all } y \in P\}$ of maximum elements forms an antichain, that intersects every maximal chain of P.

So if P has maximum chain size M, then $P \setminus A$ has maximum chain size at most $M - 1$ (in fact equal).

By induction, find a partition of $P \setminus A$ into $M - 1$ antichains and extend it by A to get a partition of P into M antichains. □

Min-max statement for max-antichains

Dilworth's Theorem

$\max\{|A| : A \text{ is an antichain}\} =$
$= \min\{|\mathcal{C}| : \mathcal{C} \text{ is a chain partition of } P\}$

Proof. (Tverberg) $\boxed{\leq}$ is again immediate.

$\boxed{\geq}$ If there is a chain, that intersects every maximal antichain of P, then we proceed by induction as in the Proposition.
Otherwise let C be a maximal chain and $A = \{a_1, \ldots, a_M\}$ be an antichain of maximum size such that $A \cap C = \emptyset$.
Let

$$\begin{aligned} A^- &= \{x \in P : x \leq a_i \text{ for some } i\} \\ A^+ &= \{x \in P : x \geq a_i \text{ for some } i\} \end{aligned}$$

- $A^- \cap A^+ = A$ because A is antichain
- $A^- \cup A^+ = P$ because A is maximal.

Apply induction on A^- and on A^+.

For this note that

$A^- \neq P \;\Leftarrow\; \max C \in A^+ \setminus A \;\Leftarrow\; C$ is maximal
$A^+ \neq P \;\Leftarrow\; \min C \in A^- \setminus A \;\Leftarrow\; C$ is maximal

Obtain

a chain partition $C_1^-, \ldots, C_M^-$ of A^- and
a chain partition $C_1^+, \ldots, C_M^+$ of A^+, such that
$C_i^- \cap A = \{a_i\} = C_i^+ \cap A$ for all i.

Then $C_1^- \cup C_1^+, \ldots, C_M^- \cup C_M^+$ is a partition of P into M chains. □