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# *In Class Sheet -1*

**Problem 1** [In class]
Show that the sum of the first $n$ integers equals the number of fields $(i, j)$ in a $n \times n$ lattice that remain if you cut away from the $n^2$ fields those with $i > j$.

**Problem 2** [In class]
How many zeroes does the the number $n!$ finish with?

**Problem 3** [In class]
Given is a chess board $B$ of size $2^n \times 2^n$ where one arbitrary field is cut out. Show that one can perfectly tile the remaining fields by dominoes with an L-form, i.e. dominoes that can cover for instance the fields $(1, 1)$, $(1, 2)$, $(2, 1)$.

**Problem 4** [In class]
The following "theorem" is obviously wrong. Read the proof in detail and find the mistake in it!

**Satz 1.** *Given are $n$ lines in the plane such that no pair of lines is parallel. Then all the lines are intersecting in one point.*

*Beweis.* The statement is true for one line and also for two, since the lines are not parallel by assumption. Assume now the statement is true for an arbitary number of $n - 1$ line (induction hypothesis). In the following we show that the theorem is then also valid for $n$ lines.
Let $S = \{a, b, c, d, \ldots\}$ be the set of the $n$ lines in the plane that are not parallel to each other. If we delete line $c$ then we get a subset $S'$ of $n - 1$ lines. From the induction hypothesis follows that the lines in $S'$ intersect in one point. Let $P$ be this point. In particular $a$ and $b$ are going through $P$, so $P$ must be the intersection of $a$ and $b$.
Now we put line $c$ back into the set $S'$ and remove another line $d$. Again we obtain a subset $S''$ of $n - 1$ lines and we can again conclude that all lines in $S''$ intersect in a point $P'$. As before, $P'$ must be the intersection point of the lines $a$ and $b$. So $P = P'$. As we have seen, the line $c$ is also contains $P$, so all lines from $S$ intersect in $P$ which proves the theorem. $\square$

**Problem 5** [In class]
Where is the mistake in the following proof?

**Satz 2.** $n^3 \leq 3n^2$ *holds for all positive integers.*

*Beweis.* We prove the statement by induction in $n$. The statement is obviously true for $n = 1$, because $1^3 \leq 3 \cdot 1^2$. Moreover it holds for $n = 1$ that

$$3n^2 \leq 3n + 2.$$

Let now $n = k + 1$. Because of the induction basis we can assume that $k^3 \leq 3k^2$ and $3k^2 \leq 3k + 2$ are holding. We add $3k + 1$ on both sides of the second inequality (which doesn't change anything because $k \geq 0$) and add that inequality then to the first one. We obtain:

$$k^3 + 3k^2 + 3k + 1 \leq 3k^2 + 3k + 2 + 3k + 1.$$

The left hand side is nothing else than $(k+1)^3 = n^3$, the right hand side is $3(k+1)^2 = 3n^2$ which proves the statement. $\square$

**Problem 6** [In class]

Given is a set of $m = 2^k$, $k \in \mathbb{N}$, lights which are sitting around a round disc that can be turned into any one of $m$ positions. Each light has a switch that can turn the light on and off, independently of the others. A game between Alice and Bob runs as follows. Bob switches some of the lights on. In one round of the play, first Alice chooses a set of positions. Then Bob turns the disc by an angle of $2\pi i/m$ ($i \in \mathbb{N}$) of his choice and flips the switches of the lights at the positions Alice originally had chosen.
Prove that Alice has a strategy that turns off all the lights.