email: andreas.loos@math.fu-berlin.de

Exercise sheet 12

Problem 57 [10 points]

Let T a graph. We define

$$\mathrm{rad}(T) := \min_{v \in V(T)} \big(\max_{u \in V(T) \setminus \{v\}} (\mathrm{dist(u, v)}) \big)$$

The *center* of T is the set of vertices in V such that there is a vertex $u \in V$ with $\mathrm{dist(u, v)} = \mathrm{rad(T)}$.

(a) Prove that the center of a tree T is either a vertex or an edge in T.

(b) Show that every automorphism of a tree fixes a vertex or an edge.

Problem 58 [10 points]

Prove that the number of spanning trees of the graph G on vertices $\{1, 2, \ldots, n\}$ obtained from the complete graph by deleting an edge is $n^{n-3}(n-2)$.

Problem 59 [10 points]

Given is a graph G and two spanning trees $T_1, T_2 \subseteq G$. Show that for all $e_1 \in E(T_1) \setminus E(T_2)$ there is an edge $e_2 \in E(T_2) \setminus E(T_1)$ such that $T_1 - e_1 + e_2$ and $T_2 - e_2 + e_1$ are *both* spanning trees.

Problem 60 [10 points]

Two players play a game on a graph G, alternatingly choosing distinct vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together the players follow a path. The last player able to move wins.

Prove that the second player has a winning strategy if and only if G has a perfect matching.

Problem 61 [10 points]

Let G be a k-regular bipartite multigraph. Show that the edges of G can then be partitioned into k perfect matchings. Can you show the same if every vertex has degree $\leq k$?