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Exercise sheet 13

This is a practice sheet for the material of the next-to-last week.

Problem 62
Show that in any graph G the size of any maximal matching is at least $\alpha'(G)/2$.

Problem 63
Let G a bipartite graph $G = (A \cup B, E)$ which has a matching M of size $|A|$. Prove that there is a vertex v in A such that all edges edges incident to v belong to a maximum size matching. *Hint:* Consider a set $X \subseteq A$ with the property that $|N(X)| = |X|$ but for every subset $S \subset\neq X$, $|N(S)| > |S|$, and prove that every vertex $v \in X$ has the desired property.

Problem 64 [10 points]
(a) (*Polygamy Hall Theorem*) Given a bipartite graph $H = (A \cup B, E)$ such that for every $S \subseteq A$ $|N(S)| \geq 2|S|$, show that there exists a family of pairwise disjoint subgraphs isomorphic to $K_{1,2}$ such that every vertex of A is the midpoint of one.
(b) (*Generalizing Tic-Tac-Toe*) A *positional game* consists of a set $X = \{x_1, \ldots, x_n\}$, the *board*, and designated subsets $W_1, \ldots, W_m \subseteq X$ of the board, the *winning sets.* (Traditional 3×3 Tic-Tac-Toe has a board with nine elements and eight winning sets: the horizontal, vertical and diagonal lines.) Two players alternately choose elements of X; a player wins by choosing all elements of a winning set first.
Suppose that each winning set has size at least 10 and each element of the board appears in at most 5 winning sets. Prove that Second Player can force at least a draw.(*Hint:* Show that Second Player can find a family of disjoint pairs of elements of the board such that each winning set contains at least one of these pairs and explain how he could use such a pairing to draw the game. (Such a strategy is called a *pairing strategy.*))

Problem 65
Let G be a connected graph in which for every edge e, there are cycles C_1 and C_2 containing e whose only common edge is e. Prove that G is 3-connected. Use this to show that the Petersen graph is 3-edge-connected.

Problem 66
Let G be a k-connected graph. Define a graph $G' \supseteq G$ by adding a new vertex $v \notin V(G)$ to $V(G)$ and making it adjacent to k vertices in $V(G)$. Prove that G' is also k-connected.

Problem 67
For every $k, l, m \in \mathbb{N}$, $k \leq l \leq m$, construct a graph G with $\kappa(G) = k$, $\kappa'(G) = l$, $\delta(G) = l$.