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Exercise sheet 13 — Solutions

This is a practice sheet for the material of the next-to-last week.

Problem 62
Show that in any graph G the size of any maximal matching is at least $\alpha'(G)/2$.

This is about the difference between the concepts of a *maximal matching* and *maximum matching*. Recall that a maximal matching is a matching that can not be enlarged by adding another edge. A maximum matching is a matching that has maximal cardinality among all matchings of G; we denote this cardinality by $\alpha'(G)$. Let M be a maximal matching. Then every edge of G contains at least one of the $2|M|$ vertices M saturates (In other words, the union of the edges of M is a vertex cover of G). In particular, if $e_1, e_2, \ldots, e_{\alpha'(G)}$ is a maximum matching, then, since these edges are pairwise disjoint, they each contain (at least) one *different* vertex from the $2|M|$ vertices saturated by M. Therefore $\alpha'(G) \leq 2|M|$.

Problem 63
Let G a bipartite graph $G = (A \cup B, E)$ which has a matching M of size $|A|$. Prove that there is a vertex v in A such that all edges incident to v belong to a maximum size matching.
(*Hint:* Consider a set $X \subseteq A$ with the property that $|N(X)| = |X|$ but for every subset $S, \emptyset \subsetneq S \subsetneq X$, we have $|N(S)| > |S|$, and prove that every vertex $v \in X$ has the desired property.)

Let $X \subseteq A$ be a minimal non-empty subset with $|N(X)| \geq |X|$ (Since there is a matching saturating A, all subsets have this property). Then for every $S, \emptyset \subsetneq S \subsetneq X$, we have $|N(S)| > |S|$.
Let M be a matching saturating A. Since $|N(X)| = |X|$, M has $|X|$ edges between X and $N(X)$. Hence the sub-matching $M_1 \subseteq M$ saturating $A \setminus X$ does not saturate any vertex in $N(X)$.
Let $vw \in E(G)$ for $v \in X$. We extend M_2 to a matching of size $|A|$ that contains vw. Let $G' = G - v - w$ and $X' := X \setminus \{v\}$. For every subset $S \subseteq X'$ in G' we have $|N_{G'}(S)| \geq |N_G(S)| - 1 \geq |S|$, since $|N_G(S)| > |S|$ holds for every nonempty $S \subsetneq X$ by assumption. Therefore, by Hall's theorem there is a matching M' saturating X' in G'. Then $M' \cup \{vw\} \cup M_1$ is a matching with cardinality $|A|$ in G.

Problem 64

(a) (*Polygamy Hall Theorem*) Given a bipartite graph $H = (A \cup B, E)$ such that for every $S \subseteq A$ $|N(S)| \geq 2|S|$, show that there exists a family of pairwise disjoint subgraphs isomorphic to $K_{1,2}$ such that every vertex of A is the midpoint of one.

(b) (*Generalizing Tic-Tac-Toe*) A *positional game* consists of a set $X = \{x_1, \dots, x_n\}$, the *board*, and designated subsets $W_1, \dots, W_m \subseteq X$ of the board, the *winning sets*. (Traditional 3×3 Tic-Tac-Toe has a board with nine elements and eight winning sets: the horizontal, vertical and diagonal lines.) Two players alternately choose elements of X; a player wins by choosing all elements of a winning set first. Suppose that each winning set has size at least 10 and each element of the board appears in at most 5 winning sets. Prove that Second Player can force at least a draw.(*Hint:* Show that Second Player can find a family of disjoint pairs of elements of the board such that each winning set contains at least one of these pairs and explain how he could use such a pairing to draw the game. (Such a strategy is called a *pairing strategy*.))

(a) Create an auxiliary bipartite graph H by adding a new vertex v' for each vertex $v \in A$ and edges $v'w$ to every neighbor $w \in N(v)$ of v. Formally let $A' = \{v' : v \in A\}$ and $E(H) = E(G) \cup \{v'w : vw \in E(G)\}$.
We check Hall's condition for H. For a subset $S \subseteq A \cup A'$ suppose, without loss of generality, that $|S \cap A| \geq |S \cap A'|$, that is $|S \cap A| \geq \frac{1}{2}|S|$. Then by our condition on G we have

$$|N_H(S)| \geq |N_G(S \cap A)| \geq 2|S \cap A| \geq |S|.$$

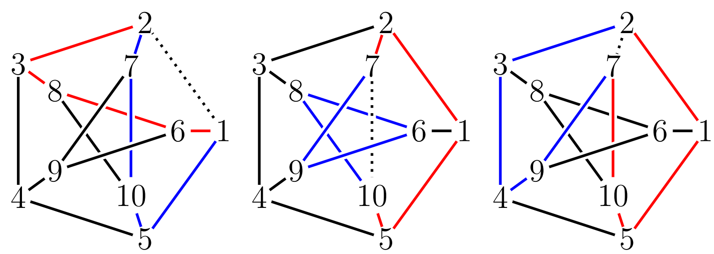
Therefore by Hall's theorem, there is a matching $M = \{uw_u; u \in A \cup A'\}$ in H saturating $A \cup A'$. Then for $v \in A$ the subgraphs in G with vertex set $w_v, v, w_{v'}$ are pairwise disjoint and isomorphic to $K_{1,2}$.

(b) Let $\mathcal{W} = \{W_1, \dots, W_m\}$ be the set of the winning sets. We construct an auxiliary bipartite graph H with node set $\mathcal{W} \dot{\cup} X$ and edge set $E(H) = \{W_i x_j : x_j \in W_i\}$.
We check the Polygamy Hall condition and hence derive that there is a matching in H saturating $\mathcal{W}$. Let $\mathcal{S} \subseteq W$. We count the edges of H between $\mathcal{S}$ and its neighborhood $N(\mathcal{S})$. On the one hand the degree of every vertex in $\mathcal{S}$ is at least 10, so at least $10|S|$ edges are leaving $\mathcal{S}$ and of course all enter $N(\mathcal{S})$. On the other hand, the degree of the vertices in X at most 5, so there are at most $5|N(\mathcal{S})|$ edges entering $N(\mathcal{S})$ (from anywhere). Hence $10|\mathcal{S}| \leq 5|N(\mathcal{S})|$, that is $2|\mathcal{S}| \leq |N(\mathcal{S})|$. So we can apply a) and find for each winning set $W_i \in \mathcal{W}$ a pair $X_i \subseteq W_i$ such that these pairs are pairwise disjoint.
Given these X_i, the strategy goes as follows. Player 2 makes sure that in every pair X_i he has at least one of the two elements and hence he occupies an element in every winning set W_i. This is easy to do: if Player 1 chooses an element from X_i, then Player 2 immediately chooses the other. (This is possible to do, since the X_i are pairwise disjoint.) If Player 1 chooses an element outside of $\cup_{i=1}^m X_i$ then Player 2 chooses an arbitrary free element.

Problem 65

Let G be a connected graph in which for every edge e, there are cycles C_1 and C_2 containing e whose only common edge is e. Prove that G is 3-connected. Use this to show that the Petersen graph is 3-edge-connected.

Let $uv \in E(G)$ be part of a minimum edge cut $[S, \overline{S}]$ Then the nodes u and v must be on the different sides of the edge cut.
The edge $e = uv$ is in two cycles C_1 and C_2 with $(C_1 \setminus e) \cap (C_2 \setminus e) = \varnothing$. Each of these cycle must then have another edge $e_1 \neq uv$, $e_2 \neq uv$ in $[S, \overline{S}]$ and these edges are distinct. Therefore the minimum edge cut has at least three edges, so G is 3-edge-connected.
Now consider the following image of the Petersen graph. There are three classes of edges: edges in the outer cycle, edges in the inner cycle and edges connecting outer and inner cycle. For each class of edges one can find two disjoint cycles. Because of the symmetry, it suffices to show the existence for one example in each class:



Problem 66
Let G be a k-connected graph. Define a graph $G' \supseteq G$ by adding a new vertex $v \notin V(G)$ to $V(G)$ and making it adjacent to k vertices in $V(G)$. Prove that G' is also k-connected.

Let G be a k-connected graph. Let G' be a graph obtained from G by adding a new vertex v with at least k neighbours.
Assume for a contradiction that there is a vertex cut S of size at most $k-1$ in G'.

Case 1: If $v \notin S$ then $G'[V \setminus S] = G - S$ is still connected since there is no vertex cut of size at most $k-1$ in G. The vertex v still has at least one neighbour left in $G - S$, so the whole $G' - S$ is connected.

Case 2: If $v \in S$ then $S \setminus \{v\}$ is a vertex cut of size $k-2$ of G, which is a contradiction.

So there is no vertex cut of size at most $k-1$ in G', that is, G' is k-connected.

Problem 67
For every $k, l, m \in \mathbb{N}$, $k \leq l \leq m$, construct a graph G with $\kappa(G) = k$, $\kappa'(G) = l$, $\delta(G) = l$.

Let $0 < k \leq l \leq m$ be integers. We want to construct a graph G with vertex-connectivity $\kappa(G) = k$, edge-connectivity $\kappa'(G) = l$ and minimum degree $\delta(G) = m$. Start with two disjoint copies of K_{m+1} on vertex sets V_1,V_2. Choose two sets $A = \{a_1, \ldots, a_k\} \subseteq V_1$ and $B = \{b_1, \ldots, b_k\} \subseteq V_2$ of k vertices and connect them using l edges such that all edges of the form $a_i b_i$ are there (and the remaining $l - k$ edges are arbitrary between A and B).
The degree of every vertex is at least m, since every vertex is contained in one of the K_{m+1}. Furthermore, since $k < m + 1$, there is a vertex which is not contained in any of the l crossing edges, so the minimum degree is exactly m.

Either of the two sets A,B is a vertex cut of size k, so the connectivity is at most k. Assume there is a vertex cut S of size $k-1$. After removing S both K_{m+1}'s remain connected. Moreover since $|S| = k-1$ there exists an $i \in [k]$ such that $a_i \notin S$ and $b_i \notin S$, so the edge $a_i b_i$ connects the remainder of the two cliques. Thus there is no vertex cut of size $k-1$ and hence the connectivity is exactly k.
The edge connectivity is at most l, since $[V_1, \overline{V_1}]$ is an edge cut of size l. For any other edge cut $[S, \bar{S}]$ there exists an $i \in \{1, 2\}$ such that $\emptyset \neq S \cap V_i \neq V_i$ and therefore $|[S, \bar{S}]| \geq |S \cap V_i|(m + 1 - |S \cap V_i|) \geq m \geq l$, so the edge-connectivity is exactly l.