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Exercise sheet 14

This is a practice sheet for the material of the last week.

Problem 62
Show that for any graph G, we have $\chi(G) + \chi(\bar{G}) \leq v(G) + 1$.

Problem 63
Prove that if G is k-color-critical, then the Mycielski graph $M(G)$ of G is $(k+1)$-color-critical.

Problem 64
Prove that minimum number of edges a k-chromatic graph can have is $\binom{k}{2}$. Use this to prove that if G is contained in the union of m copies of K_m then $\chi(G) \leq m^{3/2}$.

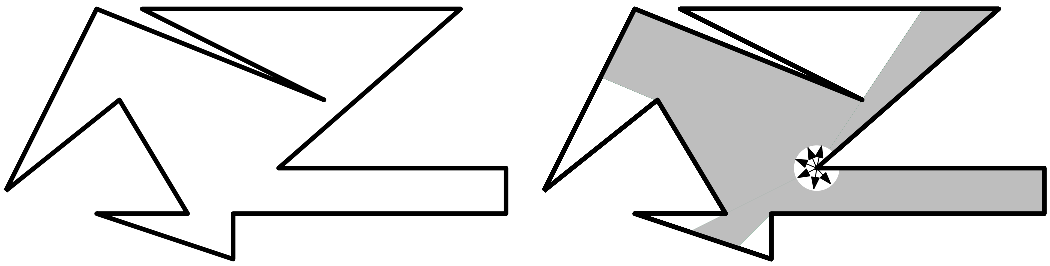
Remark: The Erdős-Faber-Lovász Conjecture (1972) states that if the m copies of K_m are *pairwise edge-disjoint* then $\chi(G) = m$. Paul Erdős offered $ 500 for a proof. (after Erdős' death the pledges he made for the resolution of various mathematical problems are still honored by Ron Graham.)

Definition A planar graph G is *outerplanar* if there is an embedding of it in the plane such that all vertices are on the boundary of the outer face.

Problem 65
Use Kuratowski's Theorem to show that a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

Problem 66
Prove, without using the Four Color Theorem, that every outerplanar graph is 3-colorable.

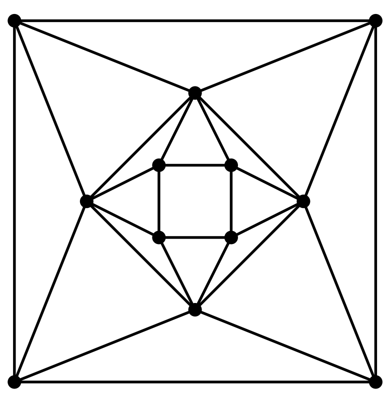
Problem 67
Apply Problem 2 to prove the Art Gallery Theorem: If an art gallery is laid out as a simple polygon with n sides, then it is possible to place $\lfloor n/3 \rfloor$ guards such that every point of the interior is visible by some guard.
Construct a polygon that does require $\lfloor n/3 \rfloor$ guards.



An art gallery and what a guard sees from a corner

Problem 68
Prove that every simple planar graph with at least four vertices has at least four vertices with degree less than 6.
For each even value of n with $n \geq 8$, construct an n-vertex simple planar graph G that has exactly four vertices with degree less than 6.

Problem 69
Define a sequence of plane graphs as follows. Let $G_1 = C_4$. For $n > 1$ obtain G_n from G_{n-1} by adding a new 4-cycle surrounding G_{n-1}, making each vertex of the new cycle also adjacent to two consecutive vertices of the previous outside face. The graph G_3 is shown below.
Prove that if n is even, then every proper 4-coloring of G_n uses each color on exactly n vertices.



Problem 70

(a) Give a drawing of K_6 in the real projective plane without any crossing. (Think of the projective plane as a closed disc where opposite points of the boundary circle are identified.)

(b) Give a drawing of K_7 on the torus without any crossing. (Think of the torus as the unit square $[0,1]^2$, where each boundary point $(0,y)$ is identified with $(1,y)$ and point $(x,0)$ is identified with $(x,1)$.)