**Exercise 62**

We prove the statement by induction on the number $n$ of vertices in $G$. If $n = 1$ then $G = K_1$ and $\chi(G) + \chi(\overline{G}) = 1 + 1$, so the base case is fine.

Let $n > 1$. Take an arbitrary vertex $v \in V(G)$, delete it from $G$ and apply induction for $G' = G - v$. By definition $\overline{G - v} = \overline{G} - v$, so

$$\chi(G - v) + \chi(\overline{G} - v) \leq n - 1 + 1 = n.$$

Clearly, $\chi(G) \leq \chi(G - v) + 1$ and $\chi(\overline{G}) \leq \chi(\overline{G} - v) + 1$, since one could always create a proper coloring of $G$ (or $\overline{G}$) by taking an optimal coloring of $G - v$ (or $\overline{G} - v$) and add a new color to $v$. So

$$\chi(G) + \chi(\overline{G}) \leq \chi(G - v) + 1 + \chi(\overline{G} - v) + 1 \leq n + 2.$$

Since all the numbers involved are integers, if *any* of the inequalities above are strict, we proved the required upper bound of $n+1$. Hence we are done otherwise $\chi(G) = \chi(G-v)+1$ and $\chi(\overline{G}) = \chi(\overline{G}-v)+1$, as well as $\chi(G-v)+\chi(\overline{G}-v) = n$.

Note, however that if indeed $\chi(H) = \chi(H - u) + 1$ for some graph $H$ and vertex $u \in V(H)$, then $d(u) \geq \chi(H - u)$, since if $|N(u)| \leq \chi(H - u) - 1$ then an optimal coloring of $H - u$ would be possible to extend to $u$ by simply taking any color which does not appear on the neighbors of $u$.

This observation provides contradiction with the three equalities above. From the first two equations we get $d_G(v) \geq \chi(G - v)$ and $d_{\overline{G}}(v) \geq \chi(\overline{G} - v)$, implying

$$n - 1 = d_G(v) + d_{\overline{G}}(v) \geq \chi(G - v) + \chi(\overline{G} - v)$$

and contradicting the third equation.

**Exercise 63**

A graph $G$ is $k$-colour-critical if $\chi(H) < \chi(G) = k$ for every proper subgraph $H$ of $G$. Let $M(G)$ be the Mycielski of a $k$-colour-critical graph $G$ on vertex set $V(G) = \{v_1, \dots, v_n\}$. That is a graph with $V(M(G)) = V(G) \cup \{u_1, \dots, u_n, w\}$ and $E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}\}$. We know from the lecture that $\chi(G) = k$ implies $\chi(M(G)) = k + 1$.

Since there are no isolated vertices in $M(G)$, it is enough to check that $M(G) - e$ is $k$-colorable for every edge $e \in E(M(G))$. There are three cases.

*Case 1:* $e = v_i v_j$ for some $1 \leq i < j \leq n$. Since $G$ is color-critical, we can color $G - e$ properly with $k - 1$ colors, say 1 up to $k - 1$. Then, we color the vertices $u_1, \dots, u_n$ with color $k$, and color $w$ with color 1. This is a proper $k$-coloring of $M(G) - e$.

*Case 2:* $e = v_i u_j$ for some $1 \leq i \neq j \leq n$. By the definition of Mycielski's construction, we have $v_i v_j \in E(G)$. Now, consider $H = G - v_i v_j$. Since $G$ is $k$-color-critical, $H$ is $(k-1)$-colorable. So, $M(H)$ is $k$-colorable by the theorem in the lecture. Moreover, we can see that $M(G) - e = M(H) + v_i v_j + v_j u_i$.

Now, the idea is to color $M(H)$ first by $k$ colors properly, and modify this coloring into a proper $k$-coloring of $M(G) - e$. Here is an explicit method. First we color $V$ by $k - 1$ colors, say 1 up to $k - 1$, so that this will be a proper $(k - 1)$-coloring of $H$. Then, for each $\ell \in \{1, \dots, n\}$, color $u_\ell \in U$ by the color used for $v_\ell \in V$. Finally, we color $w$ by the color $k$. This is a proper $k$-coloring of $M(H)$.

Now, we add $v_iv_j$ and $v_ju_i$ to $M(H)$ so that the result will be $M(G) - e$. Then, we change the color of $v_j$ to the color $k$. Since the color $k$ was not used in $U \cup V$, this coloring is still proper. Thus, we obtained a proper $k$-coloring of $M(G) - e$.

*Case 3:* $e = u_iw$ for some $i = 1, \dots, n$. First, consider the graph $G - v_i$. Since $G$ is color-critical, $G - v_i$ is $(k-1)$-colorable. Now, in $M(G) - e$, we color $V \setminus \{v_i\}$ by $k - 1$ colors, say 1 up to $k - 1$, according to a proper $k - 1$-coloring of $G - v_i$. Next, for each $\ell \in \{1, \dots, n\} \setminus \{i\}$ we color $u_\ell$ by the color used for $v_\ell$. Then, we can color $v_i, u_i, w$ by the color $k$. We can see that this is a proper $k$-coloring of $M(G) - e$ because $v_i, u_i, w$ form an independent set in $M(G) - e$, and the color $k$ is not used on the other vertices.

To summarize, in each of the three cases, we have obtained a proper $k$-coloring of $M(G) - e$, showing that Mycielski's construction preserves color-criticality.

**Exercise 64**

Let $G$ be a $k$-chromatic graph. Take a proper $k$-colouring of $G$. For every pair of colours $i$ and $j$ there exists an edge with adjacent vertices coloured with $i$ and $j$. Indeed, otherwise we could combine the vertices of colour $i$ and $j$ into a single colour class, resulting in a proper $(k-1)$-colouring, which is in contradiction with $\chi(G) = k$. There are $\binom{k}{2}$ possible pairs of distinct colours, giving us at least $\binom{k}{2}$ edges in $G$.

Let $G$ be contained in the union of $m$ copies of $K_m$ (not necessarily edge- or vertex-disjoint). This implies $e(G) \leq m\binom{m}{2}$. Let $k$ be the chromatic number of $G$. Then by the above $\binom{k}{2} \leq e(G)$. Putting the two inequalities together we obtain $\binom{k}{2} \leq m\binom{m}{2}$, which is equivalent to $k^2 - k < m^3 - m^2$. This implies $k^2 < m^3$.

**Exercise 65**

Kuratowski's Theorem says that a graph is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$.

Let $G$ be an outerplanar graph and fix an embedding such that all vertices are on the boundary of the outer face. Construct a graph $G'$ by adding a new vertex $v$ in the outer face and connecting it to all vertices of $G$. $G'$ is planar and thus does not contain a subdivision of $K_5$ or $K_{3,3}$. Hence $G$ does not concatin a subdivision of $K_4$ or $K_{2,3}$, because adding $v$ would construct a subdivision of $K_5$ or $K_{3,3}$ in $G'$.

Let $G$ be a graph not containing a subdivision of $K_4$ or $K_{2,3}$. Again we construct $G'$ by adding a vertex $v$ and connecting it to all vertices of $G$. Now $G'$ does not contain a subdivision of $K_5$ or $K_{3,3}$, so it is planar and we can consider an embedding of $G'$ without intersections of the edges. If $v$ does not lie on the boundary of the outer face, we can apply a map onto the sphere and back onto the plane to make one of the faces next to $v$ the outer face. If we remove $v$ then all vertices are on the boundary of the outer face, because they were connected to $v$ in an embedding without intersections of edges and thus $G$ is outerplanar.

**Exercise 66**

We proove the statement by induction on the number of vertices $n$. For $n = 1, 2, 3$ every graph is 3-colourable. Let $G$ be an outerplanar graph. If we have more than one connected component, apply the induction hypothesis and 3-colour each component. Otherwise fix and embedding of $G$ such that all vertices are on the boundary of the outer face.

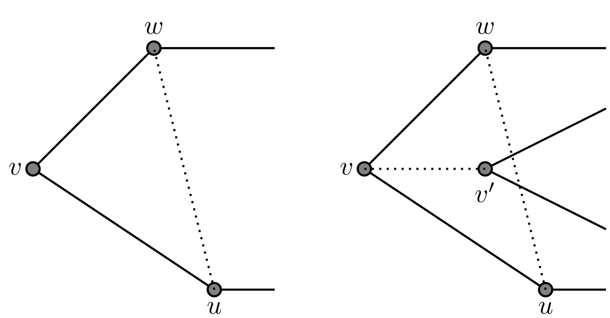
Let $v$ be an arbitrary vertex. Since we have at least 4 vertices the degree of $v$ is at least 2. If it is 2 we remove $v$ and connect both neighbours. We get an embedding of an outerplanar graph with one vertex less. So we can apply the induction hypothesis and 3-colour $G'$. To get a colouring for $G$ we colour $v$ by the third colour not used by both neighbours, which is therefore proper.

If the degree of $v$ is at least 3, let $u$ be one of the neighbours which is not next to $v$ on the boundary of the outer face. We split $G$ into two outerplanar graphs both containing $v$ and $u$ with smaller number of vertices and 3-colour both by induction hypothesis. W.l.o.g. $v$ and $u$ have the same colour in both graphs and this way we get a proper colouring of $G$ because there are no further edges between both parts.

**Exercise 67**

Take a simple (without holes) polygon $P$ with $n \geq 3$ sides and vertices. A diagonal is a non intersecting line segment connecting two vertices of $P$ which is fully contained inside $P$. A maximal set of non intersecting diagonals is called a triangulation of $P$.

First we prove that there always exists a triangulation. For $n = 3$ we only have one triangle. For $n > 3$ we need to find one diagonal. Let $v$ be the leftmost (according to $x$-coordinate) vertex of $P$ and $u$,$w$ both neighbours. If the line segment form $u$ to $w$ is a diagonal we can triangulate the rest by induction.

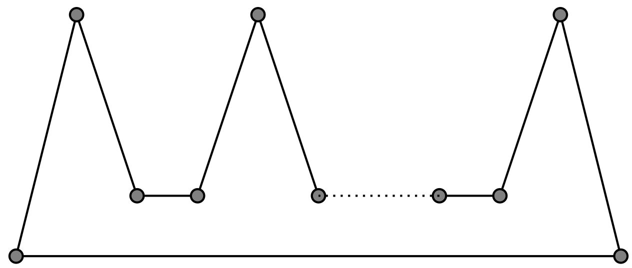



Otherwise let $v'$ be the leftmost vertx inside the triangle $uvw$, then the line segment from $v$ to $v'$ is a diagonal and we are also done.

We proved implicitly that in a triangulation all internal faces are triangles, because otherwise we can apply the induction step to any non triangular face, giving us a new diagonal contradicting the maximality.

Now we can view the triangulated polygon $P$ as an embedding of a graph $G$ into the plane with all vertices on the boundary of the outer face, i.e. $G$ is outerplanar. By Exercise 2 we know that $G$ is 3-colourable. Every internal face of the embedding is a triangle, so it has a vertex of every colour. Thus every colourclass is a valid set of guards, especially the smallest class of size $\lfloor n/3 \rfloor$.

This is an example for a polygon achieving this bound.



**Exercise 68**

Let $G$ be a simple planar graph with $n \geq 4$ vertices. We add edges to $G$, until for all $e \notin E(G)$, $G + e$ is not planar. If this graph has 4 vertices with degree less than 6, then the original graph too.

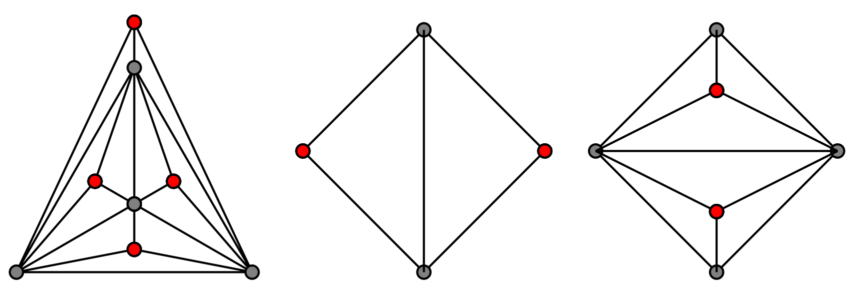
Now every vertex in $G$ has degree at least 3. If there would be a 0 or 1 degree vertex we could immediately add an edge still leaving the graph planar. If there is a vertex of degree 2 there are two adjacent faces. Since we have at least 4 vertices at most one of them can be a triangle giving us the possibility to add another edge.

Let us assume that we only have 3 vertices with degree less that 6, then with $e$ being the number of edges in $G$ we have 3 vertices of degree at least 3 and $n-3$ vertices of degree at least 6, thus

$$2e \geq (n-3) * 6 + 3 * 3 = 6n - 9.$$

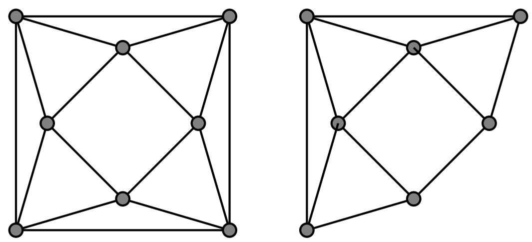
We know from the Corollary of Euler's Theorem that in a planar graph with $n \geq 3$ we have $m \leq 3n - 6$. Together this gives us a contradiction and thus $G$ has at least 4 vertices of degree less than 6.

The constructions are below. On the left is the case $n = 8$ and on the right the step from $n$ to $n + 2$. The four red vertices are of degree 3, all other have degree 6. We can always add two vertices keeping the number of red vertices four, because the structure shown in the middle always reconstructs.



**Exercise 69**

Since $G_n$ is planar we know by the Four Colour Theorem that there exists a proper colouring of $G_n$. If we look at any two consecutive 4-cycles in $G_n$ they form a subgraph which is isomorphic to $G_2$. So it suffices to proof the statement for $G_2$. $G_2$ is 4-chromatic but not 4-colour-critical, because after removing any vertex it remains 4-chromatic. Thus we still need 4 colours for $G_2 - v$ and therefore every colour exactly twice in any colouring of $G_2$.



**Exercise 70**

Drawings for $K_6$ on sphere and $K_7$ on torus.