email: andreas.loos@math.fu-berlin.de 12/14 May 2012

Exercise sheet 3

Due **2PM, Friday, 8 May 2015**
in the mailbox of Andreas Loos (Villa Arnimallee 2) or via e-mail

Problem 11 [10 points]
Let $p(n;k,m)$ be the number of ways to partition n into k parts with largest part m (and $p(0;0,0)$ is defined to be 1). Show that for every $K, M \geq 0$,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{K} \sum_{m=1}^{M} p(n;k,m) = \binom{K+M}{K}.$$

Problem 12 [10 points]
Prove that for for Stirling numbers of the first kind it holds that

$$s_{n+1,k+1} = \sum_{i=0}^{n} \binom{i}{k} s_{n,i}.$$

Problem 13 [10 points]
Recall that $p(n) = \sum_{i=0}^{n} p(n,i)$ denotes the number of partitions of n in any number of parts. Show that for $n \geq 2$,

$$p(n) - p(n-1)$$

equals the number of partitions of n in which the two largest parts are equal.

Problem 14 [10 points]
Show that on average, permutations of length n have H_n cycles, where H_n denotes the nth Harmonic number

$$H_n := \sum_{i=1}^{n} \frac{1}{i}.$$

(*Hint:* You might want to try double counting.)

Problem 15 [10 points]
Show that the number of possibilities to distribute n pennies to piles such that each pile contains an *odd number* of pennies is equal to the number of possibilities to distribute the pennies such that each pile contains a *different number* of pennies.

Problem 16 [10 points]

(a) Show that for binomial coefficients $\binom{n}{k}$ holds that

$$\binom{n}{0} < \binom{n}{1} < \ldots < \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil} > \ldots > \binom{n}{n},$$

where for even n the two middle coefficients are the same.

(b) Show a similar result for Stirling numbers of the first kind. For every $n \geq 1$ there is a $M(n)$, such that either

$$s_{n,0} < s_{n,1} < \ldots < s_{n,M(n)} > s_{n,M(n)+1} > \ldots > s_{n,n}$$

or

$$s_{n,0} < s_{n,1} < \ldots < s_{n,M(n)-1} = s_{n,M(n)} > \ldots > s_{n,n},$$

hold, where $M(n) = M(n-1)$ or $M(n) = M(n-1) + 1$. Hint: Use the recursions for $s_{n,k}$ to construct a proof by induction.

Remark 1. The analogous statement also holds for Stirling numbers of the second kind $S_{n,k}$.