Discrete Mathematics 1 — Summer Term 2015

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Exercise sheet 4

Due **2PM, Friday, 15 May 2015**
in the mailbox of Andreas Loos (Villa Arnimallee 2) or via e-mail

Problem 17 [10 points]

Prove the following connection between binomial coefficients and Fibonacci numbers. For every $n \geq 0$,

$$F_n = \sum_{k=0}^{n} \binom{n-1-k}{k}.$$

(Bonus: Can you also give a combinatorial solution by considering the family of subsets of $[n]$ which do not contain two consecutive integers?)

Problem 18 [10 points]

Find a closed form of the generating function for the "swapped Fibonacci" sequence, that is the sequence you obtain from the Fibonacci sequence by swapping F_{2j} with F_{2j+1} for every $j = 0, 1, 2, 3, \ldots$. (The start of the swapped Fibonacci sequence is $1, 1, 3, 2, 8, 5, 21, 13, \ldots$.)

Problem 19 [10 points]

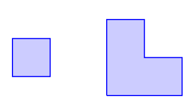
Find the generating function of the sequence $(a_0, a_1, \ldots)$ given by the recurrence relation $6a_n = 3a_{n-1} - 2a_{n-2} + a_{n-3}$ for every $n \geq 3$ and the initial values $a_0 = 14$, $a_1 = 2$, and $a_2 = 14$ and use it to derive a closed formula for a_n.

Problem 20 [10 points]

Show that $(7 + \sqrt{50})^{2015}$ has at least 2015 zeroes following the decimal point.
(*Hint:* Try to set up a recurrence relation the solution of which is close to this number in question.)

Problem 21 [10 points]

Write down a recurrence relation for the sequence c_n, where c_n represents the number of ways one can cover a $n \times 2$-rectangle completely without overlap with the following two types of tiles?



The tiles my be rotated by integral multiples of 90 degree.
Do NOT solve the recurrence! Make sure your case analysis is well-argued for!

Problem 22 [10 points]

Finish the proof of the theorem in the lecture about homogeneous linear recurrences. Let k be a positive integer and let

$$p(x) = x^k - \alpha_{k-1}x^{k-1} - \cdots - \alpha_1 x - \alpha_0$$

be a polynomial where $\alpha_0, \dots, \alpha_{k-1} \in \mathbb{C}$ and $\alpha_0 \neq 0$[1]. Let $\lambda_1, \dots, \lambda_q \in \mathbb{C}$ be the distinct roots of $p(x)$, with multiplicity $k_1, \dots, k_q$, respectively. That is, $k_1 + \cdots k_q = k$ and

$$p(x) = (x - \lambda_1)^{k_1} (x - \lambda_2)^{k_2} \cdots (x - \lambda_q)^{k_q}.$$

Show that for every sequence $(a_0, a_1, \dots)$ satisfying the recurrence

$$a_n = \alpha_{k-1} a_{n-1} + \cdots \alpha_0 a_{n-k} \text{ for all } n \geq k$$

there exist constants $C_{ij} \in \mathbb{C}$ for every $i = 1, \dots q$ and $j = 0, \dots k_i - 1$, such that for every integer $n \geq 0$ we have

$$a_n = \sum_{i=1}^{q} \sum_{j=0}^{k_i - 1} C_{ij} n^{\underline{j}} \lambda_i^{n-j}.$$

(*Hint:* Note that a root λ of a polynomial has multiplicity at least 2 if and only if λ is also root of the derivative.)

[1]Note that $\alpha_0 \neq 0$ is not a real restriction: the coefficient of the last term a_{n-k} of the recurrence can always be assumed to be non-zero, otherwise we have a recurrence with fewer terms.