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Exercise sheet 6

Due **2PM, Friday, 29 May 2015**
in the mailbox of Andreas Loos (Villa Arnimallee 2) or via e-mail

Problem 27 [10 points]
Rank the following functions according to growth rate using the „big-O“ measures

$$
\begin{array}{rrrr}
n & n! & (\log_2 n)^{\log_2 n} & \log_4 n \\
\log_3 n & n^2 & 101 & n\log_2 n \\
n^2\log_2 n & n^{1+\frac{1}{\log_2 n}} & \log_3^2 n & \log_2 n^5
\end{array}
$$

Express the ordering using asymptotic notation such that for each "neighbouring" pair of functions you determine the asymptotic notation (O, Θ, o, or $\sim$) that captures their relationship the closest.
For this problem you do not have to give any proofs.

Problem 28 [10 points]
Show that from $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$ follows that

$$f_1(n) + f_2(n) = O\left(g(n)\right).$$

where $g(n) = \max\{g_1(n), g_2(n)\}$

Problem 29 [10 points]
The function $\pi(n)$ counts the prime numbers up to n, that is,

$$\pi(n) := |\{p \in [n] : \ p \text{ is a prime}\}|$$ [1]

(a) Show that every prime number p, $m < p \leq 2m$, divides $\binom{2m}{m}$.

(b) Show that $\pi(n) = O\left(\frac{n}{\ln n}\right)$.

Problem 30 [10 points]

(a) Show that if p^k is a prime power that divides $\binom{2m}{m}$, then $p^k \leq 2m$.

(b) Show that $\pi(n) = \Omega\left(\frac{n}{\ln n}\right)$.

[1] The growth rate of $\pi(n)$ fascinated mathematicians for centuries, before finally in 1896 Hadamard and de la Vallée Poussain proved that $\pi(n) \sim \frac{n}{\ln n}$. This means that on the average roughly every $\ln n$ integer is a prime up to n. In the first two exercises you are asked to show a weaker statement, that the *order of magnitude* of $\pi(n)$ is $\frac{n}{\ln n}$. Even though these arguments might seem relatively simple in retrospect, it was only after many decades of unsuccesful tries by such greats as Gauss and Legendre that Chebyshev found a proof in 1852.

Problem 31 [10 points]

Let there be a voting between two candidates A and B with A being the winner. Each of the n voters had one vote which he expressed by putting a black or white ball into a ballot box. The counting committee pulls out the balls one by one and counts them.

(a) How many possibilities are there to count these balls such that in every phase A has more votes than B?

(b) Suppose the counting takes place in a TV show and is therefore supposed to be as exciting as possible. How many possibilities are there to count the balls in such a way that the *time* while it appears that B is winning is as large as possible?

Problem 32 [10 points]

What is the number of squares that have as vertices at least two gridpoints of a rectangular $k \times k$-grid, if the sides of the squares are always parallel to the gridlines?