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Exercise sheet 8

Due **2PM, Friday, 12 June 2015**
in the mailbox of Andreas Loos (Villa Arnimallee 2) or via e-mail

Problem 38 [10 points]
Let $x_1, \dots, x_{2n}$ be real numbers with $|x_i| \geq 1$ for all i, and let $I \subset \mathbb{R}$ be an arbitrary open interval of length 2. Prove that the number of sums $\sum_{k=1} \epsilon_k x_k$, where the ϵ_k are ± 1, which fall in the interior of I does not exceed $\binom{2n}{n}$.
Show that for a closed interval I of length 2 the statement is not necessarily true.
Bonus: How should the upper bound change so the statement stays true with closed intervals?

Problem 39 [10 points]
Show that any poset with n elements contains either a chain or a an antichain of at least $\sqrt{n}$ elements.

Problem 40 [10 points]
Let P and Q be finite posets and let $P \times Q$ their direct product. Show that if $(s,t) \leq (s',t')$ in $P \times Q$, then

$$\mu_{P\times Q}\big((s,t),(s',t')\big) = \mu_P(s,s')\mu_Q(t,t').$$

Problem 41 [10points]
(a) Show that for every $n \in \mathbb{N}$ we have

$$\sum_{\substack{d\in[n]\\ d|n}} \varphi(d) = n.$$

(*Hint:* Classify the elements of $[n]$ according to their g.c.d. with n.)

(b) Use (a) and number theoretic Möbis Inversion to reprove the formula $\varphi(n) = n\prod_{p|n}(1-\frac{1}{p})$ we have derived in the lecture.

Problem 42 [10 points]
Let b_n be the number of cyclic 0/1 sequences of length n. For example, $b_1 = 2, b_2 = 3$, and $b_3 = 4$. Furthermore, let a_n be the number of cyclic 0/1 sequences of length n that are aperiodic. (A cyclic sequence of length n is called *aperiodic* if the shortest rotation that brings it back to itself is the one with 360 degrees). For example, $a_1 = 2, a_2 = 1$, and $a_3 = 2$.
(a) Prove that $b_n = \sum_{d|n} a_d$
(b) Prove that $2^n = \sum_{d|n} d \cdot a_d$
(c) Prove that $a_n = \frac{1}{n}\sum_{d|n} \mu(\frac{n}{d})2^d$
(d) Derive $b_n = \frac{1}{n}\sum_{d|n} 2^d\varphi(\frac{n}{d})$ and calculate b_{12}.