Discrete Mathematics I — So 2016
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Exercise Sheet 0

Do date: Apr 19th/20th, 10:15 AM, during the exercise class.
Not to be submitted.

This sheet contains some practice problems to help get you warmed up for this semester's course. You are invited to work on them together and discuss your solutions during the exercise classes of the first week. Your solutions will not be graded.

Exercise 1 In a biology experiment, a hungry rat is put in the bottom-left corner of a 4×4 grid, while some tasty cheese[1] is placed in the top-right corner.

(a) Since the rat is hungry[2] and clever[3], it only moves to up or to the right. How many different routes are there for the rat to reach the cheese?

(b) This rat is actually very hungry, so sometimes it takes shortcuts by making diagonal moves, moving up and right in the same step. Now how many different ways are there for it to reach the cheese?

Exercise 2 How many zeroes does the number 50! end with?

Exercise 3 Find the flaws, if any exist, in the following proof.

Claim: All horses are the same colour.[4]

Proof: We prove the following statement by induction on n: any set of n horses all have the same colour.

For the base case of $n = 1$, consider any set $\{H_1\}$ of a single horse. Since there is only one horse, there is only one colour, and the statement holds true.

[1]While it would appear that rodents are naturally lactose-intolerant, we will pay homage to the common cartoon cliché that they all love cheese.

[2](because the scientists starved it)

[3](which is what the scientists are trying to observe)

[4]If you were not friends with very many horses while growing up, a quick Google[5]search will show you that this is not true, and so either there should be a flaw in the proof, or this proof actually proves that mathematics should not be applied to real-world situations.

[5]No sponsorship was paid for this mention of Google (sadly). If Bing or Yahoo or DuckDuckGo would like to pay me a small fee, I would gladly cite them instead.

For the induction step, suppose the statement is true for n. Now take any $n+1$ horses $H_1, H_2, \ldots, H_n$ and H_{n+1}. If we consider the subset $\{H_1, H_2, \ldots, H_n\}$, we have a collection of n horses. By the induction hypothesis, these all have the same colour. In particular, all of these horses are the same colour as H_2. Now consider the subset $\{H_2, \ldots, H_n, H_{n+1}\}$. This is another set of n horses, and so by the induction hypothesis, all of these horses are the same colour. Again, they must all be the same colour as the horse H_2. This shows that every horse in $\{H_1, \ldots, H_{n+1}\}$ has the same colour as H_2, and in particular has the same colour, proving the induction step. $\square$

Exercise 4 Find the flaws, if they exist, in the following proof.

Claim: For any integer $n \geq 0$, $2^n = 1$.

Proof: We prove this by strong induction on n.

For the base case of $n = 0$, we have $2^0 = 1$ as claimed.

For the induction step, we wish to prove $2^{n+1} = 1$, and may assume $2^k = 1$ for all $k \leq n$. This follows from a bit of algebraic manipulation, using the induction hypothesis in the third equality:

$$2^{n+1} = \frac{2^{2n}}{2^{n-1}} = \frac{2^n \cdot 2^n}{2^{n-1}} = \frac{1 \cdot 1}{1} = 1.$$

Hence, by induction, we have $2^n = 1$ for all integers $n \geq 0$. $\square$

Exercise 5 Prove that for any $n \geq 1$, if any square in a $2^n \times 2^n$ grid is removed, then it can be perfectly tiled[6] by L-shaped tiles — that is, tiles of the shape $\{(1,1), (1,2), (2,1)\}$.

Exercise 6 You and 99 of your friends enter a dark room, excited to take part in the newest "Escape" game in Berlin. That is the last thing you remember, though, when you wake up in a brightly-lit dungeon. Confused, you hear the only door being unlocked, and see someone walk in.

> "Welcome to Dungeon Escape," she says. "As you can see, while we drugged you and brought you to this secret location, we put hats on your heads. Each hat is either red or blue. You can see everyone else's hat, but you cannot see your own." You look around, and find that 37 of your friends have red hats, and the other 62 of them have blue hats.
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> "In order to escape the room, you need to figure out what colour your own hat is. Of course, you cannot just take it off and check." As she says this, you realise that your hands are handcuffed behind your back, and you are chained out of reach of all of your friends, indeed making it impossible to reach your hat.
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> "You also cannot talk to each other — you must figure this out on your own. We will be watching you," she warns, as she points out the several video cameras

[6] That is, every remaining square is covered without any overlaps.

> in this dungeon, "and if there is any illegal communication, you will be stuck here forever." Given the drugging and the handcuffs, you have no doubt that she is serious about this threat, and you can see that all of your friends also feel the same way. There will not be any communication.
>
> "Every minute I will come back and open the door. If any of you have figured out what colour your hat is, you can tell me. If you are right, you are free to leave. If you are wrong though ... ." Her voice trails off, but you understand that you cannot afford to be wrong. Nobody will guess the colour of their hat until they have full confidence in their answer.
>
> "A quick hint to get you started: at least one of you has a red hat. See you in a minute!" So saying, she laughs and locks the door behind her as she leaves.

Of course, since you see 37 red hats, you already knew her 'hint'. In fact, although you do not realise this, you have a red hat as well, making it a 38/62 red/blue split.

The hundred of you are very eager to get out of this dungeon as quickly as possible, and leave a very negative review on Yelp[7]. Luckily for you, you are all also very smart, and will reach any logical conclusions immediately.

How long will it take you to realise that you have a red hat, and thus escape the dungeon?

[7] Again, no advertising fees were collected here.